How instanton combinatorics solves Painlevé VI, V and III's

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Abstract. We elaborate on a recently conjectured relation of Painlevé transcendents and 2D CFT. General solutions of Painlevé VI, V and III are expressed in terms of $c = 1$ conformal blocks and their irregular limits, AGT-related to instanton partition functions in $\mathcal{N} = 2$ supersymmetric gauge theories with $N_f = 0, 1, 2, 3, 4$. Resulting combinatorial series representations of Painlevé functions provide an efficient tool for their numerical computation at finite values of the argument. The series involve sums over bipartitions which in the simplest cases coincide with Gessel expansions of certain Toeplitz determinants. Considered applications include Fredholm determinants of classical integrable kernels, scaled gap probability in the bulk of the GUE, and all-order conformal perturbation theory expansions of correlation functions in the sine-Gordon field theory at the free-fermion point.

1. Introduction

Painlevé transcendents [13] are nowadays widely recognized as important special functions with a broad range of applications including integrable models, combinatorics and random matrix theory. Many aspects of Painlevé equations, such as their analytic and geometric properties, asymptotic problems, special solutions and discretization, have been extensively studied in the last four decades.

From the point of view of the theory of classical special functions [58], the surprising feature of these developments is the absence of transparent connection to representation theory. Instead, the Riemann-Hilbert approach [22] is typically used. It is well-known that Painlevé equations emerge most naturally in the study of monodromy preserving deformations of linear ODEs. Thus, by analogy with the solution of classical integrable systems by the inverse scattering method, the questions on nonlinear Painlevé functions

may be asked in terms of linear monodromy. In particular, one may attempt to realize the following program:

- label different Painlevé functions by monodromy data of the auxiliary linear problem,
- express their asymptotics near the critical points in terms of monodromy,
- construct full solution using the asymptotic behaviour as initial condition.

Starting from the foundational work of Jimbo [31], there are many results available on the first two points, but the lack of algebraic structure makes the last one difficult to tackle. In other words, the question is

> *...how does one combine asymptotic information about the solutions obtained from the Riemann-Hilbert problem, together with efficient numerical codes in order to compute the solution $u(x)$ at finite values of x?* [14, Painlevé Project Problem].

In [27], a solution of this problem was suggested for the sixth Painlevé equation. It was shown that Painlevé VI tau function $\tau_{\mathrm{VI}}(t)$ can be thought of as a correlation function of primary fields in 2D conformal field theory [5] with central charge $c = 1$. Under natural minimal assumptions on primary content of the theory and fusion rules, $\tau_{\mathrm{VI}}(t)$ may then be written as a linear combination of Virasoro conformal blocks. Being purely representation-theoretic quantities, these CFT special functions can be computed in several ways. In particular, the recently proven [1] AGT conjecture [2] relates them to instanton partition functions in $\mathcal{N} = 2$ SUSY 4D Yang-Mills theories [6, 21, 43, 44], which can be expressed as sums over tuples of partitions. This results into combinatorial series representations of $\tau_{\mathrm{VI}}(t)$ around the critical points $0, 1, \infty$.

The aim of this note is to extend the results of [27] to Painlevé V and Painlevé III, and to make them accessible to a wider audience. With this purpose in mind, we deliberately include some background material and illustrate our claims with a number of explicit examples and applications to random matrix theory and integrable QFT.

The plan is as follows. Section 2 sets the notation and explains the relation between different Painlevé equations and their various forms. In Section 3, we recall some basics on conformal blocks and AGT correspondence. Conjectural general solutions of Painlevé VI, V and III are presented and discussed in Section 4. In particular, it is shown that our combinatorial expansions can be seen as a generalization of Gessel's theorem representation of classical Toeplitz determinant solutions. Section 5 is devoted to applications, which include Fredholm determinants of classical integrable kernels (hypergeometric, Whittaker, confluent hypergeometric and modified Bessel), scaled GUE bulk gap probability and correlators of exponential fields in the sine-Gordon model at the free-fermion point.

2. Painlevé equations

2.1. Conventional form

Painlevé VI, V, and III (P_{VI}, P_{V}, P_{III}) first appeared as a part of the classification of 2nd order, 1st degree nonlinear ODEs without movable critical points. In this context, they are usually written as follows:

Painlevé VI:

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} + \\ + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right), \tag{2.1}$$

Painlevé V:

$$\frac{d^2q}{dt^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}, \tag{2.2}$$

Painlevé III:

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q}. \tag{2.3}$$

It is often convenient to use instead of P_{III} an equivalent equation,

Painlevé III′:

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2(\alpha + \gamma q)}{4t^2} + \frac{\beta}{4t} + \frac{\delta}{4q}, \tag{2.4}$$

which reduces to P_{III} by setting $t_{\mathrm{III}'} = t^2_{\mathrm{III}}$, $q_{\mathrm{III}'} = t_{\mathrm{III}}q_{\mathrm{III}}$.

2.2. Parameterization

We write four P_{VI} parameters as

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{VI}} = \left(\left(\theta_\infty + \frac{1}{2}\right)^2, -\theta_0^2, \theta_1^2, \frac{1}{4} - \theta_t^2\right). \tag{2.5}$$

If $\delta \neq 0$ in P_{V}, then one can set $\delta = -\frac{1}{2}$ by rescaling the independent variable. P_{V} with $\delta = 0$ is reducible to P_{III} (see e.g. transformations (1.24)–(1.26) in [12]) which will be treated separately. Hence we may set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{V}} = \left(2\theta_0^2, -2\theta_t^2, 2\theta_* - 1, -\frac{1}{2}\right). \tag{2.6}$$

The case of P_{III} is slightly more involved. In the generic situation, when $\gamma\delta \neq 0$, one can assume that $\gamma = -\delta = 4$ by rescaling t and q. We will then write

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_1} = (8\theta_\star, 4 - 8\theta_*, 4, -4). \tag{2.7}$$

The variable change $q \to q^{-1}$ maps P_{III} with $\delta = 0$ to P_{III} with $\gamma = 0$. Assume that $\gamma = 0$ and $\alpha\delta \neq 0$, then the scaling freedom can be used to set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_2} = (8, 4 - 8\theta_*, 0, -4)\,. \tag{2.8}$$

For $\gamma = \delta = 0$, $\alpha\beta \neq 0$ we can set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_3} = (8, -8, 0, 0)\,. \tag{2.9}$$

Finally, for $\alpha = \gamma = 0$ (and, similarly, for $\beta = \delta = 0$ by $q \to q^{-1}$), the general (two-parameter) solution of P_{III} is known [36]. It reads

$$q(t) = \mu t^{1-\nu} + \frac{\beta}{\nu^2}\, t + \frac{\beta^2 + \nu^2\delta}{4\mu\nu^4}\, t^{1+\nu},$$

where μ, ν are two arbitrary integration constants. Excluding this last solvable case, there remain three inequivalent P_{III}'s with two, one and zero parameters. Significance of the degenerate equations P_{III_2} and P_{III_3} was realized in [49] from a geometric viewpoint, and later they were extensively studied in [46].

2.3. Hamiltonian form

Painlevé equations can be written as non-autonomous hamiltonian systems [37]. In this approach, (2.1), (2.2) and (2.4) are obtained by eliminating momentum p from the equations

$$\frac{dq}{dt} = \frac{\partial H_{\mathrm{J}}}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{J}}}{\partial q}, \qquad\qquad \mathrm{J} = \mathrm{VI}, \mathrm{V}, \mathrm{III}'_{1,2,3},$$

where the Hamiltonians are given by

$$t(t-1)H_{\mathrm{VI}} = q\,(q-1)\,(q-t)\,p\left(p - \frac{2\theta_0}{q} - \frac{2\theta_1}{q-1} - \frac{2\theta_t - 1}{q-t}\right) + \\ + (\theta_0 + \theta_t + \theta_1 + \theta_\infty)\,(\theta_0 + \theta_t + \theta_1 - \theta_\infty - 1)\,q, \tag{2.10}$$

$$tH_{\mathrm{V}} = (q-1)(pq - 2\theta_t)(pq - p + 2\theta_*) - tpq + \left((\theta_* + \theta_t)^2 - \theta_0^2\right) q + \\ + \left(\theta_t - \frac{\theta_*}{2}\right) t - 2\left(\theta_t + \frac{\theta_*}{2}\right)^2, \tag{2.11}$$

$$tH_{\mathrm{III}'_1} = (pq + \theta_*)^2 + tp - \theta_\star q - \frac{q^2}{4}, \tag{2.12}$$

$$tH_{\mathrm{III}'_2} = (pq + \theta_*)^2 + tp - q, \tag{2.13}$$

$$tH_{\mathrm{III}'_3} = p^2 q^2 - q - \frac{t}{q}. \tag{2.14}$$

The hamiltonian structure is crucial for the construction of Okamoto-Bäcklund transformations [45], generating an infinite number of Painlevé solutions from a given one.

2.4. Sigma form and tau functions

The time-dependent Hamiltonians (2.10)–(2.14) themselves satisfy nonlinear 2nd order ODEs. To write them, introduce auxiliary functions

$$\begin{aligned}\sigma_{\mathrm{VI}} = t(t-1)H_{\mathrm{VI}} - q(q-1)p + (\theta_0+\theta_t+\theta_1+\theta_\infty)\, q \\ - (\theta_0+\theta_1)^2\, t + \frac{\theta_1^2+\theta_\infty^2-\theta_0^2-\theta_t^2-4\theta_0\theta_t}{2},\end{aligned} \tag{2.15}$$

$$\sigma_{\mathrm{J}} = tH_{\mathrm{J}}, \qquad \mathrm{J} = \mathrm{V}, \mathrm{III}'_{1,2,3}. \tag{2.16}$$

They satisfy the so-called σ-form of Painlevé equations [23, 30]:

$$P_{\mathrm{VI}}: \quad \begin{aligned}&\sigma'\Big(t(t-1)\sigma''\Big)^2 + \Big[2\sigma'(t\sigma'-\sigma) - (\sigma')^2 - (\theta_t^2-\theta_\infty^2)(\theta_0^2-\theta_1^2)\Big]^2 = \\ &= \left(\sigma'+(\theta_t+\theta_\infty)^2\right)\left(\sigma'+(\theta_t-\theta_\infty)^2\right)\left(\sigma'+(\theta_0+\theta_1)^2\right)\left(\sigma'+(\theta_0-\theta_1)^2\right),\end{aligned} \tag{2.17}$$

$$P_{\mathrm{V}}: \quad (t\sigma'')^2 = \Big(\sigma - t\sigma' + 2(\sigma')^2\Big)^2 - \frac14\Big((2\sigma'-\theta_*)^2 - 4\theta_0^2\Big)\Big((2\sigma'+\theta_*)^2 - 4\theta_t^2\Big), \tag{2.18}$$

$$P_{\mathrm{III}'_1}: \quad (t\sigma'')^2 = \left(4(\sigma')^2-1\right)(\sigma-t\sigma') - 4\theta_*\theta_\star\sigma' + \left(\theta_*^2+\theta_\star^2\right), \tag{2.19}$$

$$P_{\mathrm{III}'_2}: \quad (t\sigma'')^2 = 4(\sigma')^2(\sigma-t\sigma') - 4\theta_*\sigma' + 1, \tag{2.20}$$

$$P_{\mathrm{III}'_3}: \quad (t\sigma'')^2 = 4(\sigma')^2(\sigma-t\sigma') - 4\sigma', \tag{2.21}$$

which also appear in the classification of 2nd order, 2nd degree ODEs with Painlevé property [12].

The solutions of (2.1)–(2.4) can thus be mapped to solutions of (2.17)–(2.21). Conversely, one can recover conventional Painlevé functions from the solutions of σ-Painlevé equations using the following formulas:

$$P_{\mathrm{VI}}: \quad \begin{aligned}&\frac{1}{q-t} + \frac12\left(\frac1t + \frac{1}{t-1}\right) = \\ &= \frac{2\theta_\infty t(t-1)\sigma'' + (\sigma'+\theta_t^2-\theta_\infty^2)\left((2t-1)\sigma' - 2\sigma + \theta_0^2-\theta_1^2\right) + 4\theta_\infty^2(\theta_0^2-\theta_1^2)}{2t(t-1)\left(\sigma'+(\theta_t-\theta_\infty)^2\right)\left(\sigma'+(\theta_t+\theta_\infty)^2\right)},\end{aligned} \tag{2.22}$$

$$P_{\mathrm{V}}: \quad q = \frac{2\left(t\sigma''+\sigma-t\sigma'+2(\sigma')^2\right)}{(2\sigma'-\theta_*)^2-4\theta_0^2}, \tag{2.23}$$

$$P_{\mathrm{III}'_1}: \quad q = -\frac{2t\sigma''+4\theta_*\sigma'-2\theta_\star}{4(\sigma')^2-1}, \tag{2.24}$$

$$P_{\mathrm{III}'_2}: \quad q = -\frac{t\sigma''+2\theta_*\sigma'-1}{2(\sigma')^2}, \tag{2.25}$$

$$P_{\mathrm{III}'_3}: \quad q = -\frac{1}{\sigma'}. \tag{2.26}$$

Finally, define the tau functions of P_{VI}, P_{V} and P_{III} by

$$\sigma_{\mathrm{VI}}(t) = t(t-1)\frac{d}{dt}\ln\left(t^{\frac{\theta_0^2+\theta_t^2-\theta_1^2-\theta_\infty^2}{2}}(1-t)^{\frac{\theta_t^2+\theta_1^2-\theta_0^2-\theta_\infty^2}{2}}\tau_{\mathrm{VI}}(t)\right), \tag{2.27}$$

$$\sigma_{\mathrm{V}}(t) = t\frac{d}{dt}\ln\left(e^{-\frac{\theta_* t}{2}}t^{-\theta_0^2-\theta_t^2-\frac{\theta_*^2}{2}}\tau_{\mathrm{V}}(t)\right), \tag{2.28}$$

$$\sigma_{\mathrm{J}}(t) = t\frac{d}{dt}\ln\tau_{\mathrm{J}}(t), \qquad \mathrm{J}=\mathrm{III}'_{1,2,3}. \tag{2.29}$$

Our solution below is formulated in terms of combinatorial expansions of these tau functions in powers of t. Expansions of σ's and q's can then be obtained from the relations (2.22)–(2.26) and (2.27)–(2.29).

2.5. Coalescence

As is well-known, Painlevé VI produces all other Painlevé equations in certain scaling limits. The equations considered in the present paper form the first line of the coalescence cascade

$$\begin{tikzcd}
\mathrm{VI} \arrow[r] & \mathrm{V} \arrow[r] \arrow[d] & \mathrm{III}_1 \arrow[r] \arrow[d] & \mathrm{III}_2 \arrow[r] \arrow[d] & \mathrm{III}_3 \\
& \mathrm{IV} \arrow[r] & \mathrm{II} \arrow[r] & \mathrm{I} &
\end{tikzcd}$$

Fig. 1: Coalescence diagram for Painlevé equations

Every step to the right or to the bottom of the diagram decreases by 1 the number of parameters in the corresponding equation.

Let us now describe the scaling limits we need (1st line) and the transition $P_{\mathrm{III}_2}\to P_{\mathrm{I}}$ which seems to be missing in the literature (cf. e.g. the degeneration scheme in [46]).

- $P_{\mathrm{VI}}\to P_{\mathrm{V}}$: set in P_{VI}

$$\theta_1 = \frac{\Lambda+\theta_*}{2}, \qquad \theta_\infty = \frac{\Lambda-\theta_*}{2}, \tag{2.30}$$

then solutions of P_{V} can be obtained as the limits

$$1-q_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty}\frac{t/\Lambda}{q_{\mathrm{VI}}(t/\Lambda)}, \tag{2.31}$$

$$\sigma_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty}\left(\frac{\Lambda^2-\Lambda t-\theta_*^2-2\theta_0^2-2\theta_t^2}{4}-\sigma_{\mathrm{VI}}(t/\Lambda)\right), \tag{2.32}$$

$$\tau_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty}(t/\Lambda)^{\theta_0^2+\theta_t^2}\,\tau_{\mathrm{VI}}(t/\Lambda), \tag{2.33}$$

- $P_{\mathrm{V}}\to P_{\mathrm{III}'_1}$: this limiting transition is described by

$$\theta_0 = \frac{\Lambda-\theta_\star}{2}, \qquad \theta_t = \frac{\Lambda+\theta_\star}{2}, \tag{2.34}$$

$$q_{\mathrm{III}'_1}(t) = \lim_{\Lambda\to\infty}\Lambda\left(1-q_{\mathrm{V}}(t/\Lambda)\right), \tag{2.35}$$

$$\sigma_{\mathrm{III}'_1}(t) = \lim_{\Lambda\to\infty}\left(\frac{\Lambda^2+\theta_*^2+\theta_\star^2}{2}+\sigma_{\mathrm{V}}(t/\Lambda)\right), \tag{2.36}$$

$$\tau_{\mathrm{III}'_1}(t) = \lim_{\Lambda\to\infty}\tau_{\mathrm{V}}(t/\Lambda), \tag{2.37}$$

- $P_{\mathrm{III}_1'} \to P_{\mathrm{III}_2'}$: similarly,

$$q_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \theta_\star q_{\mathrm{III}_1'}(t/\theta_\star), \tag{2.38}$$

$$\sigma_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \sigma_{\mathrm{III}_1'}(t/\theta_\star), \tag{2.39}$$

$$\tau_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \tau_{\mathrm{III}_1'}(t/\theta_\star). \tag{2.40}$$

- $P_{\mathrm{III}_2'} \to P_{\mathrm{III}_3'}$:

$$q_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} q_{\mathrm{III}_2'}(t/\theta_*), \tag{2.41}$$

$$\sigma_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} \sigma_{\mathrm{III}_2'}(t/\theta_*), \tag{2.42}$$

$$\tau_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} \tau_{\mathrm{III}_2'}(t/\theta_*). \tag{2.43}$$

- $P_{\mathrm{III}_2'} \to P_{\mathrm{I}}$: set

$$\theta_* = 3\Lambda^{\frac{5}{4}}, \qquad t_{\mathrm{III}_2'} = 16\Lambda^{\frac{15}{4}}\left(1+\frac{t}{2\Lambda}\right), \tag{2.44}$$

$$\sigma_{\mathrm{III}_2'}\left(t_{\mathrm{III}_2'}\right) = 2\Lambda\sigma(t) + 8\Lambda^{\frac{5}{2}} + \frac{1}{4}\Lambda^{-\frac{5}{4}} t_{\mathrm{III}_2'}, \tag{2.45}$$

then in the limit $\Lambda\to\infty$ the function $\sigma(t)$ satisfies the σ-form of P_{I}, namely,

$$(\sigma'')^2 = 2\sigma - 2t\sigma' - 4(\sigma')^3. \tag{2.46}$$

Also, if (2.45) is replaced with

$$q_{\mathrm{III}_2'}\left(t_{\mathrm{III}_2'}\right) = -4\Lambda^{\frac{5}{2}} + 4\Lambda^2 q(t), \tag{2.47}$$

the limiting equation for $q(t)$ is P_{I} in the conventional form:

$$q'' = 6q^2 + t. \tag{2.48}$$

2.6. Analytic properties

The only branch points of P_{VI} and $P_{\mathrm{V,III}_{1,2,3}'}$ transcendents in the extended complex t-plane are $0, 1, \infty$ and $0, \infty$, respectively. The corresponding tau functions are holomorphic on the universal covers of $\mathbb{P}^1\backslash\{0,1,\infty\}$ and $\mathbb{P}^1\backslash\{0,\infty\}$. The functions q and σ may also have movable poles associated to zeros of τ. We introduce the branch cuts $(-\infty,0]\cup[1,\infty)$ (for P_{VI}) and $(-\infty,0]$ (for $P_{\mathrm{V,III}_{1,2,3}'}$) and adopt the principal branch convention for all fractional powers of t and $1-t$.

3. Conformal blocks and instanton partition functions

3.1. Conformal blocks

Here we review basic notions about conformal blocks in 2D CFT [5]. For the sake of brevity, simplicity and relevance for the rest of the presentation, we will concentrate on

conformal blocks for the 4-point correlator on $\mathbb{P}^1$ and Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0}.$$

Only chiral primary fields $\mathcal{O}$ will be considered, i.e. their antiholomorphic conformal dimensions $\bar{\Delta}_{\mathcal{O}} = 0$. We do not require invariance of correlators under the braid group action on the positions of fields to avoid constraints on dimensions.

The three-point correlator of primary fields is fixed by conformal symmetry up to a constant factor,

$$\langle \mathcal{O}_3(z_3)\mathcal{O}_2(z_2)\mathcal{O}_1(z_1)\rangle = C\left(\Delta_3, \Delta_2, \Delta_1\right) z_{21}^{\Delta_3-\Delta_1-\Delta_2} z_{32}^{\Delta_1-\Delta_2-\Delta_3} z_{31}^{\Delta_2-\Delta_1-\Delta_3},$$

where $z_{ij} = z_i - z_j$ and Δ_j stand for holomorphic dimensions. Thanks to conformal invariance, it suffices to consider more special coordinate dependence, namely, we may set $z_1 = 0$, $z_2 = t$, $z_3 = R$ with $R \gg t$. It is also customary to define $\langle \mathcal{O}(\infty)\ldots\rangle = \lim_{R\to\infty} R^{2\Delta_{\mathcal{O}}}\langle \mathcal{O}(R)\ldots\rangle$, so that, for instance, $\langle \mathcal{O}_3(\infty)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = C\left(\Delta_3, \Delta_2, \Delta_1\right) t^{\Delta_3-\Delta_1-\Delta_2}$.

Besides primary fields, conformal field theory also contains their descendants $L_{-\lambda}\mathcal{O} = L_{-\lambda_N}\ldots L_{-\lambda_1}\mathcal{O}$, naturally labeled by partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > 0\}$. Partitions can be identified in the obvious way with Young diagrams. As they play an important role in the rest of the paper, we take the opportunity to fix some notation for later purposes. The set of all Young diagrams will be denoted by $\mathbb{Y}$. For $\lambda \in \mathbb{Y}$, λ' denotes the transposed diagram, λ_i and λ'_j the number of boxes in ith row and jth column of λ, and $|\lambda|$ the total number of boxes. Given a box $(i,j) \in \lambda$, its hook length is defined as $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$ (see Fig. 2).

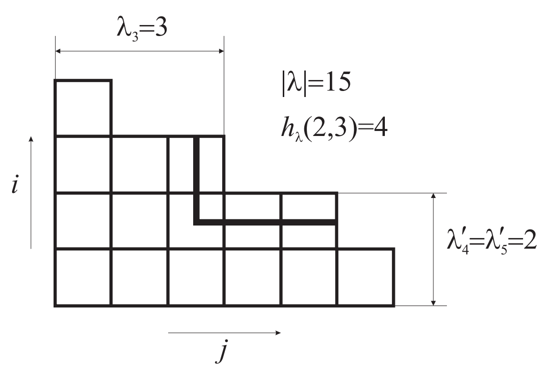

Fig. 2: Young diagram associated to the partition $\lambda = \{6, 5, 3, 1\}$

Conformal symmetry also allows to compute in explicit form the three-point functions involving one descendant:

$$\langle \mathcal{O}_3(\infty)\, \mathcal{O}_2(t) L_{-\lambda}\mathcal{O}_1(0)\rangle = C\left(\Delta_3, \Delta_2, \Delta_1\right)\gamma_\lambda\left(\Delta_1, \Delta_2, \Delta_3\right) t^{-|\lambda|}, \tag{3.1}$$

$$\langle L_{-\lambda}\mathcal{O}_3(\infty)\, \mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = C\left(\Delta_3, \Delta_2, \Delta_1\right)\gamma_\lambda\left(\Delta_3, \Delta_2, \Delta_1\right) t^{|\lambda|}, \tag{3.2}$$

where [38]

$$\gamma_\lambda\left(\Delta_1,\Delta_2,\Delta_3\right)=\prod_{j=1}^{N}\Big(\Delta_1-\Delta_3+\lambda_j\Delta_2+\sum_{k=1}^{j-1}\lambda_k\Big). \tag{3.3}$$

The action of $L_{-\lambda}$ on the field at infinity in (3.2) should be understood as a result of successive contour integration with the energy-momentum tensor. This action can be transferred to the fields at 0 and t by deformation of the contour.

The relation (3.2) is extremely important as it allows to determine the coefficients of the operator product expansion (OPE) of primary fields

$$\mathcal{O}_2(t)\mathcal{O}_1(0)=\sum_{\alpha}\sum_{\mu\in\mathbb{Y}}C\left(\Delta_\alpha,\Delta_2,\Delta_1\right)\beta_\mu\left(\Delta_\alpha,\Delta_2,\Delta_1\right)t^{\Delta_\alpha-\Delta_1-\Delta_2+|\mu|}L_{-\mu}\mathcal{O}_\alpha\left(0\right). \tag{3.4}$$

Indeed, assuming orthonormality of the basis of primaries, $\langle\mathcal{O}_\alpha\left(\infty\right)\mathcal{O}_\beta\left(0\right)\rangle=\delta_{\alpha\beta}$, and considering the correlator of both sides of the last relation with the descendant $L_{-\lambda}\mathcal{O}_\alpha\left(\infty\right)$, one finds that

$$\beta_\lambda\left(\Delta_\alpha,\Delta_2,\Delta_1\right)=\sum_{\mu\in\mathbb{Y}}\left[Q\left(\Delta_\alpha\right)\right]^{-1}_{\lambda\mu}\gamma_\mu\left(\Delta_\alpha,\Delta_2,\Delta_1\right), \tag{3.5}$$

where $Q_{\lambda\mu}\left(\Delta_\alpha\right)=\langle L_{-\lambda}\mathcal{O}_\alpha\left(\infty\right)L_{-\mu}\mathcal{O}_\alpha\left(0\right)\rangle$ is the Kac-Shapovalov matrix. It can be computed algebraically as the matrix element of descendant states

$$Q_{\lambda\mu}\left(\Delta\right)=\langle\Delta|L_{\lambda_1}\ldots L_{\lambda_N}L_{-\mu_M}\ldots L_{-\mu_1}|\Delta\rangle, \tag{3.6}$$

where $|\Delta\rangle$ and $\langle\Delta|$ denote the highest weight vectors annihilated by all $L_{n>0}$ and, respectively, all $L_{n<0}$, satisfying $L_0|\Delta\rangle=\Delta|\Delta\rangle$, $\langle\Delta|L_0=\langle\Delta|\Delta$ and normalized as $\langle\Delta|\Delta\rangle=1$. It is easy to understand that $Q(\Delta)$ has a block-diagonal structure: $Q_{\lambda\mu}\left(\Delta\right)\sim\delta_{|\lambda|,|\mu|}$.

We can now finally calculate the four-point correlator $\langle\mathcal{O}_4(\infty)\mathcal{O}_3(1)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle$. Replace therein the product of fields $\mathcal{O}_2(t)\mathcal{O}_1(0)$ by the OPE (3.4) and then use (3.1) and (3.5). The result is

$$\begin{aligned}&\langle\mathcal{O}_4(\infty)\mathcal{O}_3(1)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle=\\ &=\sum_{\alpha}C\left(\Delta_4,\Delta_3,\Delta_\alpha\right)C\left(\Delta_\alpha,\Delta_2,\Delta_1\right)t^{\Delta_\alpha-\Delta_1-\Delta_2}\mathcal{F}_c\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_\alpha;t\right),\end{aligned} \tag{3.7}$$

where we have introduced the notation

$$\mathcal{F}_c\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta;t\right)=\sum_{\lambda,\mu\in\mathbb{Y}}\gamma_\lambda\left(\Delta,\Delta_3,\Delta_4\right)\left[Q\left(\Delta\right)\right]^{-1}_{\lambda\mu}\gamma_\mu\left(\Delta,\Delta_2,\Delta_1\right)t^{|\lambda|}. \tag{3.8}$$

The representation (3.7) separates model-dependent information (three-point functions $C\left(\Delta_i,\Delta_j,\Delta_k\right)$) from the universal pieces fixed solely by Virasoro symmetry.

The function (3.8) is called *four-point conformal block*. It is a power series in t with coefficients depending on four external dimensions $\Delta_{1,2,3,4}$, one intermediate dimension Δ, and the central charge c which enters via the Kac-Shapovalov matrix. These coefficients

can in principle be calculated using (3.3) and (3.6). For the reader's convenience, we reproduce below several first terms of the series:

$$\mathcal{F}_c\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta;t\right)=1+\frac{\left(\Delta-\Delta_1+\Delta_2\right)\left(\Delta-\Delta_4+\Delta_3\right)}{2\Delta}t+$$
$$+\left[\frac{\left(\Delta-\Delta_1+\Delta_2\right)\left(\Delta-\Delta_1+\Delta_2+1\right)\left(\Delta-\Delta_4+\Delta_3\right)\left(\Delta-\Delta_4+\Delta_3+1\right)}{2\Delta\left(1+2\Delta\right)}+\right.$$
$$\left.+\frac{\left(1+2\Delta\right)\left(\Delta_1+\Delta_2+\frac{\Delta(\Delta-1)-3(\Delta_1-\Delta_2)^2}{1+2\Delta}\right)\left(\Delta_4+\Delta_3+\frac{\Delta(\Delta-1)-3(\Delta_4-\Delta_3)^2}{1+2\Delta}\right)}{\left(1-4\Delta\right)^2+\left(c-1\right)\left(1+2\Delta\right)}\right]\frac{t^2}{2}+\ldots$$

Direct (i.e. based on (3.8)) computation of conformal block coefficients becomes rather complicated at higher levels. An explicit representation for arbitrary level was found only recently in a surprisingly different framework.

3.2. $\mathcal{N}=2$ SUSY theories

The AGT correspondence [2] relates conformal blocks of 2D CFT to Nekrasov functions [43, 44]. These functions represent the instanton parts of ϵ_1,ϵ_2-regularized partition functions in 4D $\mathcal{N}=2$ SUSY quiver gauge theories.

The simplest case of AGT correspondence deals with $SU(2)$ gauge theory with extra $N_f=2N_c=4$ fundamental (i.e. transforming in the spin-$\frac12$ representation of the gauge group) matter hypermultiplets with masses $\mu_1,\ldots,\mu_4$. Parameters of this theory are related to those of 4-point conformal block on the sphere by

$$\begin{aligned}
&\mu_1=\alpha_3-\alpha_4+\frac{\epsilon}{2}, &&\mu_2=\alpha_1-\alpha_2+\frac{\epsilon}{2},\\
&\mu_3=\alpha_1+\alpha_2-\frac{\epsilon}{2}, &&\mu_4=\alpha_3+\alpha_4-\frac{\epsilon}{2},\\
&c=1+\frac{6\epsilon^2}{\epsilon_1\epsilon_2}, &&\epsilon=\epsilon_1+\epsilon_2,\\
&\Delta_\nu=\frac{\alpha_\nu\left(\epsilon-\alpha_\nu\right)}{\epsilon_1\epsilon_2}, &&\nu=1,2,3,4.
\end{aligned}$$

The intermediate dimension Δ of conformal block is expressed via the eigenvalues $\pm a$ of the vacuum expectation value of scalar field in the gauge multiplet:

$$\Delta=\frac{\alpha\left(\epsilon-\alpha\right)}{\epsilon_1\epsilon_2},\qquad \alpha=\frac{\epsilon}{2}+a.$$

The parameter t (anharmonic ratio of four points on the sphere) in the conformal block expansion is related to the bare complex coupling constant τ_{UV} on the gauge side by

$$t=\exp 2\pi i\tau_{UV},\qquad \tau_{UV}=\frac{4\pi i}{g_{UV}^2}+\frac{\theta_{UV}}{2\pi}.$$

Partition function in the regularized theory is an integral over a compactified moduli space $\mathfrak{M}$ of instantons. The integral is given by a sum of explicitly computable

contributions coming from fixed points of a torus action on $\mathfrak{M}$, which are labeled by pairs of partitions. On the CFT side, this is interpreted as an existence of a geometrically distinguished basis of states in the highest weight representations of the Virasoro algebra. For more details, generalizations and further references, the reader is referred to [4].

When all four masses $\mu_{1,2,3,4} \to \infty$, the fundamental hypermultiplets decouple and we get pure gauge theory. Decoupling only some of them yields asymptotically free theories with $N_f < 2N_c$. From the gauge theory point of view, the parameter t_{N_f} can be considered as a dynamically generated scale. Taking into account the RG dependence of the coupling constant, one finds that in the decoupling process this scale should transform in the appropriate way: $\mu_{N_f} \to \infty$, $t_{N_f} \to 0$, $t_{N_f-1} = \mu_{N_f} t_{N_f}$ fixed [16, 52]. The corresponding Nekrasov functions are related to irregular conformal blocks [25, 26, 39].

Conformal blocks relevant to Painlevé VI equation [27] are characterized by the central charge $c = 1$ and external dimensions θ_ν^2 ($\nu = 0, t, 1, \infty$) so that we can set $\epsilon_1 = -\epsilon_2 = 1$, $\alpha_1 = \theta_0$, $\alpha_2 = \theta_t$, $\alpha_3 = \theta_1$, $\alpha_4 = \theta_\infty$ and

$$\mu_1 = \theta_1 - \theta_\infty, \qquad \mu_2 = \theta_0 - \theta_t, \qquad \mu_3 = \theta_0 + \theta_t, \qquad \mu_4 = \theta_1 + \theta_\infty.$$

Under such identification of parameters, the scaling limits corresponding to the 1st line of the coalescence scheme in Fig. 1 describe successive decoupling of the matter hypermultiplets:

$$\begin{array}{ccccccccc} N_f = 4 & \xrightarrow{\mu_4 \to \infty} & N_f = 3 & \xrightarrow{\mu_3 \to \infty} & N_f = 2 & \xrightarrow{\mu_2 \to \infty} & N_f = 1 & \xrightarrow{\mu_1 \to \infty} & \text{pure gauge theory} \\ (P_{\text{VI}}) & & (P_{\text{V}}) & & (P_{\text{III}_1}) & & (P_{\text{III}_2}) & & (P_{\text{III}_3}) \end{array}$$

Fig. 3: Decoupling of matter hypermultiplets

This observation will be used in the next section for the construction of combinatorial series for P_{V} and $P_{\text{III}_{1,2,3}}$ tau functions.

4. Solutions

4.1. Painlevé VI

Let us first recall the main result of [27] as well as some motivation and evidence for it.

Conjecture 1. *Generic P_{VI} tau function can be written in the form of conformal expansion around the critical point $t = 0$:*

$$\tau_{\text{VI}}(t) = \sum_{n \in \mathbb{Z}} C_{\text{VI}}\left(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma + n\right) s_{\text{VI}}^n\, t^{(\sigma+n)^2 - \theta_0^2 - \theta_t^2} \mathcal{B}_{\text{VI}}\left(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma + n; t\right). \tag{4.1}$$

The parameters σ and s_{VI} play the role of two integration constants, $\mathcal{B}_{\text{VI}}\left(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma; t\right)$ coincides with conformal block function $\mathcal{F}_{c=1}\left(\theta_0^2, \theta_t^2, \theta_1^2, \theta_\infty^2, \sigma^2; t\right)$ and is explicitly given by combinatorial series

$$\mathcal{B}_{\text{VI}}\left(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma; t\right) = (1-t)^{2\theta_t \theta_1} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda,\mu}^{(\text{VI})}\left(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma\right) t^{|\lambda| + |\mu|}, \tag{4.2}$$

$$\mathcal{B}^{(\mathrm{VI})}_{\lambda,\mu}\left(\theta_0,\theta_t,\theta_1,\theta_\infty,\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j+\mu_i-i-j+1+2\sigma\right)^2}\times \\ \times\prod_{(i,j)\in\mu}\frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu'_j-i-j+1-2\sigma\right)^2}. \tag{4.3}$$

The structure constants in (4.1) are given by

$$C_{\mathrm{VI}}\left(\theta_0,\theta_t,\theta_1,\theta_\infty,\sigma\right)=\frac{\prod_{\epsilon,\epsilon'=\pm}G\left[\begin{array}{c}1+\theta_t+\epsilon\theta_0+\epsilon'\sigma,1+\theta_1+\epsilon\theta_\infty+\epsilon'\sigma\end{array}\right]}{\prod_{\epsilon=\pm}G\left(1+2\epsilon\sigma\right)}, \tag{4.4}$$

where $G\left[\begin{array}{c}\alpha_1,\ldots,\alpha_m\\ \beta_1,\ldots,\beta_n\end{array}\right]=\dfrac{\prod_{k=1}^m G\left(\alpha_k\right)}{\prod_{k=1}^n G\left(\beta_k\right)}$ *and* $G(z)$ *denotes the Barnes function (see Appendix A).*

The above claim was obtained in [27] by identifying $\tau_{\mathrm{VI}}(t)$ with a chiral correlator $\langle\mathcal{O}_{\mathcal{L}_\infty}(\infty)\mathcal{O}_{\mathcal{L}_1}(1)\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_0}(0)\rangle$ of Virasoro primary fields indexed by matrices $\mathcal{L}_\nu\in\mathfrak{sl}_2(\mathbb{C})$ which are related to monodromy matrices of the auxiliary linear problem for P_{VI} by $\mathcal{M}_\nu=\exp 2\pi i\mathcal{L}_\nu\in SL\left(2,\mathbb{C}\right)$. The dimensions of $\mathcal{O}_{\mathcal{L}_\nu}$ are equal to $\Delta_\nu=\frac12\mathrm{Tr}\,\mathcal{L}_\nu^2=\theta_\nu^2$.

Our main assumption is that this set of primaries closes under OPE algebra. The conservation of monodromy then implies that the dimension spectrum of primary fields appearing in the OPE of $\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_0}(0)$ is discrete and has the form $\left(\sigma_{0t}+\mathbb{Z}\right)^2$, where $2\cos 2\pi\sigma_{0t}=\mathrm{Tr}\,\mathcal{M}_0\mathcal{M}_t$. This fixes the structure of the s-channel expansion (4.1) upon identification $\sigma=\sigma_{0t}$. The constants C_{VI} are obtained from Jimbo's asymptotic formula [31] interpreted as a recursion relation, whereas (4.2)–(4.3) is nothing but the AGT representation for $c=1$ conformal block rewritten in terms of P_{VI} parameters.

In fact, Jimbo's formula also expresses the second integration constant s_{VI} in terms of monodromy. To give an explicit relation, we need to introduce monodromy invariants

$$p_\nu\ =2\cos 2\pi\theta_\nu\ =\mathrm{Tr}\,\mathcal{M}_\nu,\qquad \nu=0,1,t,\infty, \tag{4.5}$$

$$p_{\mu\nu}=2\cos 2\pi\sigma_{\mu\nu}=\mathrm{Tr}\,\mathcal{M}_\mu\mathcal{M}_\nu,\qquad \mu,\nu=0,t,1. \tag{4.6}$$

Similarly to the above, the quantities σ_{1t} and σ_{01} determine the spectrum of intermediate states in the t- and u-channel. The triple $\vec{\sigma}=\left(\sigma_{0t},\sigma_{1t},\sigma_{01}\right)$ provides the most symmetric way to label P_{VI} transcendents with the same $\vec{\theta}=\left(\theta_0,\theta_t,\theta_1,\theta_\infty\right)$. The elements of this triple are not independent: they satisfy a constraint

$$p_{0t}p_{1t}p_{01}+p_{0t}^2+p_{1t}^2+p_{01}^2-\omega_{0t}p_{0t}-\omega_{1t}p_{1t}-\omega_{01}p_{01}+\omega_4=4, \tag{4.7}$$

where

$$\omega_{0t}=p_0p_t+p_1p_\infty,$$
$$\omega_{1t}=p_tp_1+p_0p_\infty,$$
$$\omega_{01}=p_0p_1+p_tp_\infty,$$
$$\omega_4=p_0^2+p_t^2+p_1^2+p_\infty^2+p_0p_tp_1p_\infty.$$

Hence, for fixed σ_{0t}, σ_{1t} there are at most two possible values for p_{01}.

Now $s_{\rm VI}$ can be written as

$$s_{\rm VI} = \frac{(p'_{1t} - p_{1t}) - (p'_{01} - p_{01})\, e^{2\pi i \sigma_{0t}}}{(2\cos 2\pi\,(\theta_t - \sigma_{0t}) - p_0)\,(2\cos 2\pi\,(\theta_1 - \sigma_{0t}) - p_\infty)}, \tag{4.8}$$

where we have introduced the notation

$$\begin{aligned} p'_{0t} &= \omega_{0t} - p_{0t} - p_{1t}p_{01}, \\ p'_{1t} &= \omega_{1t} - p_{1t} - p_{0t}p_{01}, \\ p'_{01} &= \omega_{01} - p_{01} - p_{0t}p_{1t}. \end{aligned}$$

Remark 2. Combinatorial expansions of type (4.1) can also be found around the two remaining critical points $t = 1, \infty$, as their role is completely analogous to that of $t = 0$. For instance, the series around $t = 1$ is obtained by the exchange [31, 35]

$$t \leftrightarrow 1 - t, \qquad \theta_0 \leftrightarrow \theta_1, \qquad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p_{01} \leftrightarrow p'_{01}. \tag{4.9}$$

This gives

$$\begin{aligned} &\chi_{01}(\vec{\theta}, \vec{\sigma})\, \tau_{\rm VI}(t) = \qquad (4.10) \\ &= \sum_{n\in\mathbb{Z}} C_{\rm VI}\,(\theta_1, \theta_t, \theta_0, \theta_\infty, \sigma_{1t} + n)\, \tilde{s}^n_{\rm VI}\, (1-t)^{(\sigma_{1t}+n)^2 - \theta_t^2 - \theta_1^2}\, \mathcal{B}_{\rm VI}\,(\theta_1, \theta_t, \theta_0, \theta_\infty, \sigma_{1t} + n; 1 - t)\,, \end{aligned}$$

with $\mathcal{B}_{\rm VI}$ and $C_{\rm VI}$ defined in (4.2)–(4.4) and

$$\tilde{s}_{\rm VI} = \frac{(p'_{0t} - p_{0t}) - (p'_{01} - p_{01})\, e^{-2\pi i \sigma_{1t}}}{(2\cos 2\pi\,(\theta_t - \sigma_{1t}) - p_1)\,(2\cos 2\pi\,(\theta_0 - \sigma_{1t}) - p_\infty)}. \tag{4.11}$$

Since the normalization of $\tau_{\rm VI}(t)$ is already implicitly fixed by (4.1), the expansion (4.10) contains an additional overall constant factor $\left[\chi_{01}(\vec{\theta}, \vec{\sigma})\right]^{-1}$. Finding explicit form of this connection coefficient is an important open problem which will be treated in a separate paper. Note, however, that $\chi_{01}(\vec{\theta}, \vec{\sigma})$ disappears from $P_{\rm VI}$ functions $\sigma_{\rm VI}(t)$ and $q_{\rm VI}(t)$.

Painlevé VI equation (2.17) allows to compute the tau function expansions near the critical points recursively, order by order, starting from the leading asymptotic terms determined by Jimbo's formula. Comparing the result with Conjecture 1 provides the most straightforward and convincing test of the latter. Keeping all $\vec{\theta}$, $\vec{\sigma}$ arbitrary, we have checked in this way (see Section 3 of [27] for the details) about 30 first terms of the asymptotic expansion. Also, in a few special cases where $P_{\rm VI}$ solutions are known explicitly, the check can be carried out to arbitrary order. This includes Picard elliptic solutions and the simplest solutions of Riccati/Chazy type, which correspond to Ashkin-Teller conformal blocks [61] and correlators involving low-level degenerate fields. More complicated Riccati solutions are discussed in Subsection 4.3 of the present paper.

Numerical efficiency of the expansions (4.1), (4.10) is illustrated in Fig. 4. For random complex $\vec{\theta}$, $\vec{\sigma}$ we plot on the same graph the series for $\sigma_{\rm VI}(t)$ around $t = 0$ (blue line) and

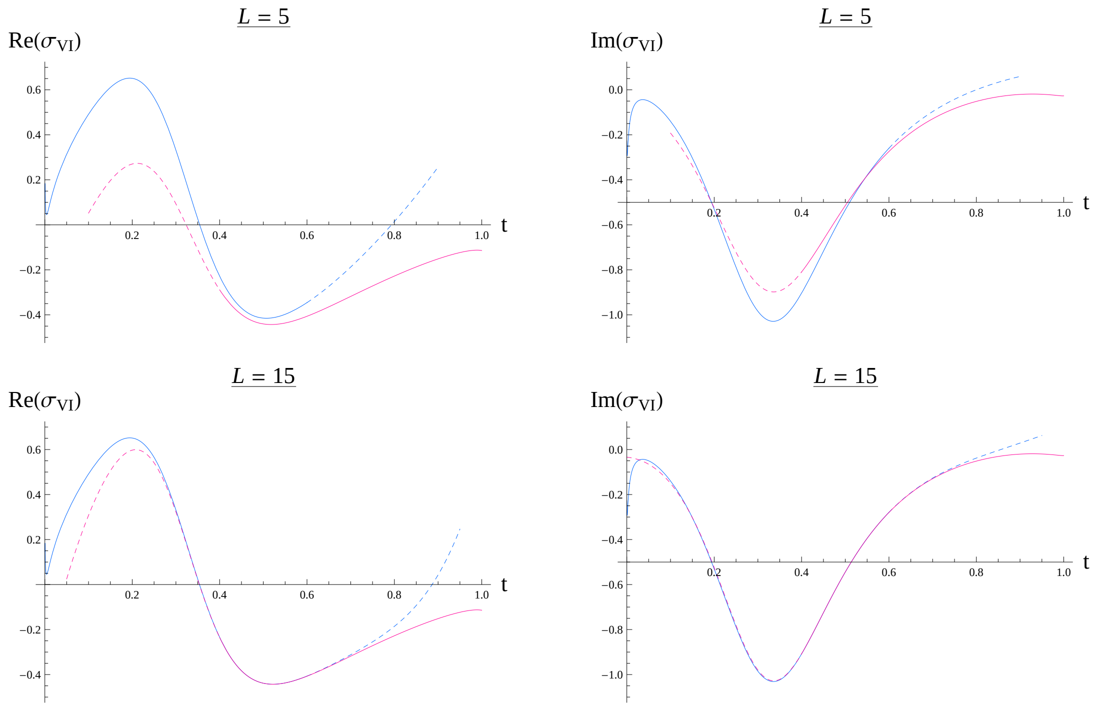

Fig. 4: Truncated P_{VI} series at $t = 0$ and $t = 1$ with
$\vec{\theta} = (0.1902 + 0.3106i, 0.4182 - 0.2109i, 0.3429 + 0.3314i, 0.0163 + 0.1805i)$,
$\vec{\sigma} = (-0.3272 - 0.4811i, 0.0958 + 0.3168i, 0.4762 + 0.1028i)$

$t = 1$ (red line) keeping the terms up to $O\left(t^L\right)$ and $O\left((1-t)^L\right)$ with $L = 5, 15$. Zooming near the endpoints $t = 0, 1$ would display oscillations of rapidly increasing frequency and decreasing amplitude due to non-zero imaginary parts of σ_{0t}, σ_{1t}.

4.2. Painlevé V and III's

Next we consider the scaling limit $P_{\mathrm{VI}} \to P_{\mathrm{V}}$ given by (2.30)–(2.33). Conformal block function $\mathcal{B}_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma; \frac{t}{\Lambda}\right)$ has a well-defined limit as $\Lambda \to \infty$, which can be calculated termwise in (4.2)–(4.3). The asymptotics of the structure constants $C_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma\right)$ ensures consistency of the expansion (4.1) with the limit (2.33). More precisely, using the estimate (A.1) from the Appendix A, it is easy to check that

$$\lim_{\Lambda\to\infty} \frac{\Lambda^{-\sigma^2} C_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma\right)}{G^2\left(1+\Lambda\right)} = C_{\mathrm{V}}\left(\theta_0, \theta_t, \theta_*, \sigma\right), \tag{4.12}$$

where

$$C_{\mathrm{V}}\left(\theta_0, \theta_t, \theta_*, \sigma\right) = \prod_{\epsilon=\pm} G\left[\begin{array}{c} 1+\theta_* + \epsilon\sigma, 1+\theta_t+\theta_0+\epsilon\sigma, 1+\theta_t-\theta_0+\epsilon\sigma \\ 1+2\epsilon\sigma \end{array}\right]. \tag{4.13}$$

One could even completely get rid of the denominator in the l.h.s. of (4.12) by modifying the normalization of P_{VI} tau function in (4.1) (e.g. by dividing all structure constants in (4.4) by a σ-independent factor $G^2(1+\theta_1+\theta_\infty)$).

Altogether, this leads to

Conjecture 3. *P_{V} is solved by the following tau function expansion at $t=0$:*

$$\tau_{\mathrm{V}}(t) = \sum_{n\in\mathbb{Z}} C_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma+n\right) s_{\mathrm{V}}^n\, t^{(\sigma+n)^2} \mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma+n;t\right). \tag{4.14}$$

Here again σ and s_{V} are arbitrary parameters, irregular conformal block $\mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma;t\right)$ is a power series defined by

$$\mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma;t\right) = e^{-\theta_t t} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{(\mathrm{V})}_{\lambda,\mu}\left(\theta_0,\theta_t,\theta_*,\sigma\right) t^{|\lambda|+|\mu|}, \tag{4.15}$$

$$\begin{aligned}\mathcal{B}^{(\mathrm{V})}_{\lambda,\mu}\left(\theta_0,\theta_t,\theta_*,\sigma\right) = & \prod_{(i,j)\in\lambda} \frac{\left(\theta_*+\sigma+i-j\right)\left(\left(\theta_t+\sigma+i-j\right)^2-\theta_0^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j+\mu_i-i-j+1+2\sigma\right)^2}\times \\ & \times \prod_{(i,j)\in\mu} \frac{\left(\theta_*-\sigma+i-j\right)\left(\left(\theta_t-\sigma+i-j\right)^2-\theta_0^2\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu'_j-i-j+1-2\sigma\right)^2},\end{aligned} \tag{4.16}$$

and the structure constants $C_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma\right)$ are given by (4.13).

The second P_{V} critical point $t=\infty$ corresponds to irregular singularity of the associated 2×2 linear system and is obtained by the fusion of two P_{VI} critical points $1,\infty$. The expansion around this point cannot be extracted from P_{VI} series and requires the knowledge of complete irregular OPEs. For the same reason, we are so far unable to treat P_{IV}, P_{II} and P_{I}. However, long-distance expansions of this kind are available in a few special cases where the solutions of $P_{\mathrm{V,III}}$ can be expressed in terms of Fredholm determinants, see Section 5.

Because of the presence of irregular singular points, monodromy data for P_{V} involve Stokes multipliers. The expression for the integration constants σ, s_{V} of Conjecture 3 in terms of monodromy can be extracted from Jimbo's paper [31].

Repeating the previous arguments almost literally for the scaling limits (2.34)–(2.37), (2.38)–(2.40) and (2.41)–(2.43), one obtains short-distance expansions for tau functions of three nontrivial P_{III} equations:

Conjecture 4. *Expansion of $\tau_{\mathrm{III}'_1}(t)$ at $t=0$ can be written as*

$$\tau_{\mathrm{III}'_1}(t) = \sum_{n\in\mathbb{Z}} C_{\mathrm{III}'_1}\left(\theta_*,\theta_\star,\sigma+n\right) s_{\mathrm{III}'_1}^n\, t^{(\sigma+n)^2} \mathcal{B}_{\mathrm{III}'_1}\left(\theta_*,\theta_\star,\sigma+n;t\right), \tag{4.17}$$

where the irregular conformal block $\mathcal{B}_{\mathrm{III}'_1}\left(\theta_*,\theta_\star,\sigma;t\right)$ is given by

$$\mathcal{B}_{\mathrm{III}'_1}\left(\theta_*,\theta_\star,\sigma;t\right) = e^{-\frac{t}{2}} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{(\mathrm{III}'_1)}_{\lambda,\mu}\left(\theta_*,\theta_\star,\sigma\right) t^{|\lambda|+|\mu|}, \tag{4.18}$$

$$\mathcal{B}^{(\mathrm{III}_1')}_{\lambda,\mu}\left(\theta_*,\theta_\star,\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\left(\theta_*+\sigma+i-j\right)\left(\theta_\star+\sigma+i-j\right)}{h_\lambda^2(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)^2}\times \\ \times\prod_{(i,j)\in\mu}\frac{\left(\theta_*-\sigma+i-j\right)\left(\theta_\star-\sigma+i-j\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)^2}, \tag{4.19}$$

and the structure constants can be written as

$$C_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma\right)=\prod_{\epsilon=\pm}G\left[\begin{array}{c}1+\theta_*+\epsilon\sigma,1+\theta_\star+\epsilon\sigma\\ 1+2\epsilon\sigma\end{array}\right]. \tag{4.20}$$

Conjecture 5. *Expansion of $\tau_{\mathrm{III}_2'}(t)$ at $t=0$ is given by*

$$\tau_{\mathrm{III}_2'}(t)=\sum_{n\in\mathbb{Z}}C_{\mathrm{III}_2'}\left(\theta_*,\sigma+n\right)s^n_{\mathrm{III}_2'}t^{(\sigma+n)^2}\mathcal{B}_{\mathrm{III}_2'}\left(\theta_*,\sigma+n;t\right), \tag{4.21}$$

with arbitrary σ, $s_{\mathrm{III}_2'}$ and

$$\mathcal{B}_{\mathrm{III}_2'}\left(\theta_*,\sigma;t\right)=\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}^{(\mathrm{III}_2')}_{\lambda,\mu}\left(\theta_*,\sigma\right)t^{|\lambda|+|\mu|}, \tag{4.22}$$

$$\mathcal{B}^{(\mathrm{III}_2')}_{\lambda,\mu}\left(\theta_*,\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\theta_*+\sigma+i-j}{h_\lambda^2(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)^2}\times \\ \times\prod_{(i,j)\in\mu}\frac{\theta_*-\sigma+i-j}{h_\mu^2(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)^2}, \tag{4.23}$$

$$C_{\mathrm{III}_2'}\left(\theta_*,\sigma\right)=\prod_{\epsilon=\pm}\frac{G(1+\theta_*+\epsilon\sigma)}{G(1+2\epsilon\sigma)}. \tag{4.24}$$

Conjecture 6. *Expansion of $P_{\mathrm{III}_3'}$ tau function at $t=0$ is:*

$$\tau_{\mathrm{III}_3'}(t)=\sum_{n\in\mathbb{Z}}C_{\mathrm{III}_3'}\left(\sigma+n\right)s^n_{\mathrm{III}_3'}t^{(\sigma+n)^2}\mathcal{B}_{\mathrm{III}_3'}\left(\sigma+n;t\right), \tag{4.25}$$

where

$$\mathcal{B}_{\mathrm{III}_3'}\left(\sigma;t\right)=\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}^{(\mathrm{III}_3')}_{\lambda,\mu}\left(\sigma\right)t^{|\lambda|+|\mu|}, \tag{4.26}$$

$$\mathcal{B}^{(\mathrm{III}_3')}_{\lambda,\mu}\left(\sigma\right) = \Big[\prod_{(i,j)\in\lambda}h_\lambda(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)\times \\ \times\prod_{(i,j)\in\mu}h_\mu(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)\Big]^{-2}. \tag{4.27}$$

$$C_{\mathrm{III}_3'}\left(\sigma\right)=\left[G\left(1+2\sigma\right)G\left(1-2\sigma\right)\right]^{-1}. \tag{4.28}$$

As in the P_{VI} case, Conjectures 3–6 can be verified by iterative reconstruction of the tau function expansions from the leading asymptotic terms using the equations $P_{\mathrm{V},\mathrm{III}'_{1,2,3}}$.

4.3. Classical solutions: AGT vs Gessel's theorem

In 2002, Forrester and Witte [24] have proved a remarkable determinant representation for a family of Riccati solutions of P_{VI}. Their result can be restated as follows. Define a five-parameter family of $N \times N$ Toeplitz determinants

$$D_N^{(\nu,\nu',\eta,\xi)}(t) = \det\left[A_{j-k}^{(\nu,\nu',\eta,\xi)}(t)\right]_{j,k=0}^{N-1}, \tag{4.29}$$

$$\begin{aligned} A_m^{(\nu,\nu',\eta,\xi)}(t) = & \frac{\Gamma(1+\nu')\, t^{\frac{\eta-m}{2}}\,(1-t)^{\nu}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+\nu')}\, {}_2F_1\left[\begin{array}{c} -\nu, 1+\nu' \\ 1+\eta-m \end{array}\Big|\frac{t}{t-1}\right] + \\ & + \frac{\xi\Gamma(1+\nu)\, t^{\frac{m-\eta}{2}}\,(1-t)^{\nu'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\, {}_2F_1\left[\begin{array}{c} 1+\nu, -\nu' \\ 1-\eta+m \end{array}\Big|\frac{t}{t-1}\right]. \end{aligned} \tag{4.30}$$

Then the function

$$\tau_N^{(\nu,\nu',\eta,\xi)}(t) = (1-t)^{-\frac{N(N+\nu+\nu')}{2}}\, D_N^{(\nu,\nu',\eta,\xi)}(t) \tag{4.31}$$

is a tau function of P_{VI} with parameters

$$(\theta_0,\theta_t,\theta_1,\theta_\infty)_{\text{VI}} = \frac{1}{2}\left(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta\right).$$

Looking at the asymptotic expansions of $D_N^{(\nu,\nu',\eta,\xi)}(t)$ at 0 and 1, one can also identify the monodromy exponents

$$(\sigma_{0t},\sigma_{1t},\sigma_{01})_{\text{VI}} = \frac{1}{2}\left(N+\eta, \nu+\nu', N+\nu-\nu'+\eta\right).$$

Almost all structure constants in (4.1) vanish because of the relations $\theta_t = \frac{N}{2}$, $\sigma_{0t} = \theta_0+\theta_t$ (recall that Barnes G-function has zeros at negative integer values of the argument). The only non-zero constants correspond to $n = 0, -1, \ldots, -N$, so that there remain only $N+1$ conformal blocks. The parameter s_{VI} in (4.1) is related to ξ in (4.30) by

$$\xi s_{\text{VI}} = \frac{\sin\pi\nu\,\sin\pi(\eta-\nu')}{\sin\pi\nu'\,\sin\pi(\eta+\nu)}. \tag{4.32}$$

Let us now consider in more detail the case $\xi \to 0$. Then (4.32) implies that $s_{\text{VI}} \to \infty$, which means that the expansion (4.1) at $t = 0$ contains only one ($n = 0$) conformal block $\mathcal{B}_{\text{VI}}\left(\frac{\eta}{2}, \frac{N}{2}, \frac{N+\nu+\nu'}{2}, \frac{\nu-\nu'+\eta}{2}, \frac{N+\eta}{2}; t\right)$. The product over boxes of μ in the AGT representation (4.3) contains a factor $i-j$ due to the relation $\sigma_{0t} = \theta_0+\theta_t$. Since this expression vanishes for the box $(1,1)$, the quantity $\mathcal{B}_{\lambda,\mu}^{(\text{VI})}$ does so for any non-empty μ. Moreover, the factor $i-j+N$ in the product over boxes of λ reduces the summation in (4.2) to Young diagrams with $\lambda_1 \leq N$ (i.e. with the length of their first row not exceeding N). Therefore, Conjecture 1 for the above parameters is equivalent to the following identity:

$$D_N^{(\nu,\nu',\eta,0)}(t) = C_N \sum_{\lambda\in\mathbb{Y}|\lambda_1\leq N} t^{|\lambda|+\frac{N\eta}{2}} \prod_{(i,j)\in\lambda} \frac{i-j+N}{i-j+N+\eta}\frac{(i-j-\nu)\,(i-j-\nu'+\eta)}{h_\lambda^2(i,j)}, \tag{4.33}$$

where the constant prefactor

$$C_N = G\left[\begin{array}{c} 1+N, 1+\nu'+N, 1+\eta, 1-\eta+\nu' \\ 1+\eta+N, 1-\eta+\nu'+N, 1+\nu' \end{array}\right] \tag{4.34}$$

can be computed using pure Fisher-Hartwig determinant.

In the limit $\eta \to 0$, the left hand side of (4.33) reduces to $N \times N$ Toeplitz determinant with the symbol

$$A(\zeta) = \left(1+\sqrt{t}\,\zeta\right)^{\nu}\left(1+\sqrt{t}\,\zeta^{-1}\right)^{\nu'}. \tag{4.35}$$

Also, $C_N = 1$ and the first factor in the product on the right disappears so that the r.h.s. coincides with the length distribution function of the first row of a random Young diagram distributed according to the so-called z-measure [8]. The equality (4.33) can then be rigorously demonstrated using a dual version of Gessel's theorem [29, 57].

Remark 7. We draw the reader's attention to the fact that Toeplitz determinant with the symbol (4.35) with $\nu = -\nu' = \frac{1}{2}$ coincides with diagonal two-point Ising spin correlation function on the infinite square lattice. Its relation to P_{VI} is rather well-known [32]. It is intriguing, however, that this lattice correlator is equal to a (particular limit of) conformal block in continuous 2D CFT with $c = 1$.

Analogous results for P_{V} and $P_{\mathrm{III}_1'}$ can be obtained by successively sending ν' and ν to infinity. For instance, consider instead of $A^{(\nu,\nu',\eta,\xi)}(t)$ and $\tau_N^{(\nu,\nu',\eta,\xi)}(t)$ the quantities

$$A_m^{(\nu,\eta,\xi)}(t) = \frac{t^{\frac{\eta-m}{2}}}{\Gamma(1+\eta-m)}\,{}_1F_1(-\nu, 1+\eta-m, -t) + \tag{4.36}$$
$$+\frac{\xi\Gamma(1+\nu)\,t^{\frac{m-\eta}{2}}e^{-t}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\,{}_1F_1(1+\nu, 1-\eta+m, t),$$

$$\tau_N^{(\nu,\eta,\xi)}(t) = t^{\frac{N^2+\eta^2}{4}}e^{\frac{Nt}{2}}\det\left[A_{j-k}^{(\nu,\eta,\xi)}(t)\right]_{j,k=0}^{N-1} \tag{4.37}$$

then $\tau_N^{(\nu,\eta,\xi)}(t)$ is a tau function of P_{V} with $(\theta_0, \theta_t, \theta_*)_{\mathrm{V}} = \frac{1}{2}(\eta, N, N+\eta+2\nu)$. Similarly, if we define

$$A_m^{(\eta,\xi)}(t) = I_{\eta-m}\left(2\sqrt{t}\right) + \xi I_{m-\eta}\left(2\sqrt{t}\right), \tag{4.38}$$

$$\tau_N^{(\eta,\xi)}(t) = t^{\frac{N^2+\eta^2}{4}}e^{-\frac{t}{2}}\det\left[A_{j-k}^{(\eta,\xi)}(t)\right]_{j,k=0}^{N-1}, \tag{4.39}$$

then $\tau_N^{(\eta,\xi)}(t)$ is a $P_{\mathrm{III}_1'}$ tau function with $\theta_* = \frac{N+\eta}{2}$, $\theta_\star = \frac{N-\eta}{2}$. For $\xi = 0$ and $\eta \to 0$ the symbols of Toeplitz determinants (4.37), (4.39) are smooth and can be written as $\left(1+\sqrt{t}\,\zeta\right)^{\nu}e^{\sqrt{t}\,\zeta^{-1}}$ and $e^{\sqrt{t}\left(\zeta+\zeta^{-1}\right)}$. Gessel representations of these determinants coincide with the results derived from Conjectures 3 and 4.

In the general case $\xi \neq 0$, the function $\tau^{(\nu,\nu',\eta,\xi)}(t)$ is a polynomial of degree N in ξ. The coefficients of $N+1$ different powers of ξ are s-channel conformal blocks

with internal dimensions $(\theta_0+\theta_t-k)^2$, where $k=0,\ldots,N$. Alternatively, one can first transform hypergeometric functions to make them depend on $1-t$ and then expand the determinant in powers of $\tilde{\xi}$, the analog of parameter ξ. The result has the form (4.10) of a sum of t-channel conformal blocks with internal dimensions $(\theta_1+\theta_t-k)^2$, again with $k=0,\ldots,N$. The relations between the expansion parameters are given by (4.32), $\tilde{\xi}\tilde{s}_{\mathrm{VI}}=\xi s_{\mathrm{VI}}=\mathcal{K}$ and $(1-s_{\mathrm{VI}})(1-\tilde{s}_{\mathrm{VI}})=1+\mathcal{K}$.

The CFT interpretation of this picture is as follows. The tau function (4.31) is a four-point correlator of primaries which involves level $N+1$ degenerate field (here $\mathcal{O}_{\mathcal{L}_t}(t)$). Its expansions at $t=0$ and $t=1$ incorporate all allowed intermediate dimensions. Determinant representation (4.29) can in fact be used to compute the fusion matrix for the corresponding two sets of conformal blocks. This task simplifies in the case $\xi=0$, where we are left with one s-channel block tranforming into a linear combination of $N+1$ t-channel ones.

5. Examples and applications

5.1. Integrable kernels

In many applications of Painlevé equations the relevant tau functions can be written as Fredholm determinants of scalar integral operators of the form $\det(1-K|_I)$, where $K|_I$ denotes the restriction of the kernel $K(x,y)$ to some interval $I\subset\mathbb{R}$. These kernels usually have integrable form, that is

$$K(x,y)=\lambda\,\frac{\varphi(x)\psi(y)-\psi(x)\varphi(y)}{x-y},\qquad \lambda\in\mathbb{C}. \tag{5.1}$$

As is well-known, given $I=\bigcup_{j=1}^{2n}(a_{2j-1},a_{2j})$ and φ, ψ verifying the differentiation formulas

$$\begin{pmatrix}\varphi'(x)\\ \psi'(x)\end{pmatrix}=A(x)\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix},$$

with some rational matrix $A(x)$, the corresponding Fredholm determinant satisfies a system of PDEs with respect to $\{a_j\}$ [56]. For φ, ψ given by classical special functions and sufficiently simple I, this system can often be solved in terms of Painlevé functions [23, 35, 54, 55, 56, 59].

5.1.1. Hypergeometric kernel. The most general known example corresponds to the choice

$$\varphi_{\mathrm{G}}(x)=\Gamma\left[\begin{matrix}1+\nu+\eta,1+\nu'+\eta'\\ 2+\nu+\nu'+\eta+\eta'\end{matrix}\right]\frac{x^{\frac{2+\nu+\nu'+\eta+\eta'}{2}}}{(1-x)^{\frac{2+\nu+\nu'+2\eta'}{2}}}\,{}_2F_1\left[\begin{matrix}1+\nu+\eta',1+\nu'+\eta'\\ 2+\nu+\nu'+\eta+\eta'\end{matrix}\Big|\frac{x}{x-1}\right],$$

$$\psi_{\mathrm{G}}(x) = \Gamma\left[\begin{array}{c} 1+\nu+\eta', 1+\nu'+\eta \\ 1+\nu+\nu'+\eta+\eta' \end{array}\right] \frac{x^{\frac{\nu+\nu'+\eta+\eta'}{2}}}{(1-x)^{\frac{\nu+\nu'+2\eta'}{2}}}\, {}_2F_1\left[\begin{array}{c} \nu+\eta', \nu'+\eta' \\ \nu+\nu'+\eta+\eta' \end{array}\Big| \frac{x}{x-1}\right],$$

with $\lambda = \pi^{-2}\sin\pi\nu\sin\pi\nu'$. The kernel $K_{\mathrm{G}}(x,y)$ contains four parameters $\nu,\nu',\eta,\eta'\in\mathbb{C}$ chosen so that the Fredholm determinant

$$D_{\mathrm{G}}(t) = \det\left(1 - K_{\mathrm{G}}\big|_{(0,t)}\right), \qquad t\in(0,1). \tag{5.2}$$

is well-defined. We will not try to determine the set of all possible values of ν,ν',η,η'; the interested reader may find examples of admissible domains in [10].

The above ${}_2F_1$ kernel first appeared in the harmonic analysis on the infinite-dimensional unitary group [10, 11]. Later it was shown [35] that the determinant (5.2) coincides with a correlator of twist fields in the massive Dirac theory on the hyperbolic disk [17, 34, 48]. From the point of view of the present paper, the most interesting feature of $D_{\mathrm{G}}(t)$ is that it is a Painlevé VI tau function, see [10] and also Sec. 5 of [35] for a simpler proof. The corresponding P_{VI} parameters are

$$(\theta_0,\theta_t,\theta_1,\theta_\infty)_{\mathrm{VI}} = \frac{1}{2}\left(\nu+\nu'+\eta+\eta', 0, \nu-\nu', \eta-\eta'\right).$$

Monodromy characterizing this particular solution is determined by [35]

$$\sigma_{0t} = \frac{\nu+\nu'+\eta+\eta'}{2}, \qquad \sigma_{1t} = \frac{\nu+\nu'}{2},$$
$$\cos 2\pi\sigma_{01} = 2e^{-\pi i(\eta+\eta'+\nu+\nu')}\sin\pi\nu\sin\pi\nu' + \cos\pi(\eta-\eta').$$

Substituting these parameters into (4.11), it can be easily checked that $\tilde{s}_{\mathrm{VI}} = 1$. Remark 2 then implies that the large gap ($t\to 1$) expansion of the ${}_2F_1$ kernel determinant is given by

$$D_{\mathrm{G}}(t) = \chi_{\mathrm{G}}^{-1}\sum_{n\in\mathbb{Z}} C_{\mathrm{G}}\left(\nu+n,\nu'+n,\eta-n,\eta'-n\right)(1-t)^{(\nu+n)(\nu'+n)}\times \tag{5.3}$$
$$\times\, \mathcal{B}_{\mathrm{G}}\left(\nu+n,\nu'+n,\eta-n,\eta'-n;1-t\right),$$

where

$$C_{\mathrm{G}}\left(\nu,\nu',\eta,\eta'\right) = G\left[1+\eta,1+\eta',1+\eta+\nu+\nu',1+\eta'+\nu+\nu'\right]\prod_{\epsilon=\pm} G\left[\begin{array}{c} 1+\epsilon\nu, 1+\epsilon\nu' \\ 1+\epsilon\left(\nu+\nu'\right) \end{array}\right],$$

$$\mathcal{B}_{\mathrm{G}}\left(\nu,\nu',\eta,\eta';1-t\right) = \mathcal{B}_{\mathrm{VI}}\left(\frac{\nu-\nu'}{2}, 0, \frac{\nu+\nu'+\eta+\eta'}{2}, \frac{\eta-\eta'}{2}, \frac{\nu+\nu'}{2}; 1-t\right),$$

and $\mathcal{B}_{\mathrm{VI}}$ is given by (4.2)–(4.3). Also, [35, Conjecture 8] suggests that the constant χ_{G} is equal to

$$\chi_{\mathrm{G}} = G\Big[1+\eta+\nu, 1+\eta+\nu', 1+\eta'+\nu, 1+\eta'+\nu'\Big]. \tag{5.4}$$

Constructing the expansion at $t=0$ is less straightforward. It is of course possible to compute a few first terms in the small gap asymptotics directly by expanding $D_{\mathrm{G}}(t)$ into Fredholm series. This yields, for instance,

$$D_{\mathrm{G}}(t) = 1 - \kappa_{\mathrm{G}} t^{1+\eta+\eta'+\nu+\nu'}\left[1+o(1)\right],$$

with

$$\kappa_{\mathrm{G}} = \lambda\, \Gamma\left[\begin{array}{c} 1+\eta+\nu, 1+\eta'+\nu, 1+\eta+\nu', 1+\eta'+\nu' \\ 2+\eta+\eta'+\nu+\nu', 2+\eta+\eta'+\nu+\nu' \end{array}\right].$$

On the other hand, direct application of Conjecture 1 is ambiguous because of special parameter values. First, Barnes functions $G\left(1+\theta_t \pm (\theta_0 - \sigma_{0t} - n)\right)$ in the structure constants vanish for $n \gtrless 0$. At the same time the quantity s_{VI} diverges due to zero denominator. The right way to handle this is to fix the values of θ's and σ_{1t}, and then consider the limit $\sigma_{0t} \to \theta_0$ with the help of the formulas (A.2)–(A.3) from the Appendix A. The result is that only the terms with $n \geq 0$ survive in the sum over n and the structure constants reduce to

$$\tilde{C}_{\mathrm{G}}(\nu,\nu',\eta,\eta',n) = (-\lambda)^n\, G\left[\begin{array}{c} 1+n, 1+\eta+\eta'+\nu+\nu'+n \\ 1+\eta+\eta'+\nu+\nu'+2n \end{array}\right]^2 \times \tag{5.5}$$

$$\times\, G\left[\begin{array}{c} 1+\eta+\nu+n, 1+\eta'+\nu+n, 1+\eta+\nu'+n, 1+\eta'+\nu'+n \\ 1+\eta+\nu, 1+\eta'+\nu, 1+\eta+\nu', 1+\eta'+\nu' \end{array}\right].$$

In addition, because of the factors $\theta_t + i - j \pm (\sigma_{0t} + n - \theta_0)$ in the products over boxes of $\lambda, \mu \in \mathbb{Y}$ combinatorial summation in conformal blocks can be restricted to Young diagrams with $\lambda_1 \leq n$, $\mu'_1 \leq n$.

This leads to the following expansion of $D_{\mathrm{G}}(t)$ near $t=0$:

$$D_{\mathrm{G}}(t) = \sum_{n=0}^{\infty} \tilde{C}_{\mathrm{G}}\left(\nu,\nu',\eta,\eta',n\right) t^{n(n+\eta+\eta'+\nu+\nu')} \sum_{\lambda,\mu\in\mathbb{Y}|\lambda_1,\mu'_1\leq n} \mathcal{B}^{\mathrm{G}}_{\lambda,\mu}\left(\nu,\nu',\eta,\eta',n\right) t^{|\lambda|+|\mu|}, \tag{5.6}$$

where $\tilde{C}_{\mathrm{G}}\left(\nu,\nu',\eta,\eta',n\right)$ is given by (5.5) and

$$\mathcal{B}^{\mathrm{G}}_{\lambda,\mu}\left(\nu,\nu',\eta,\eta',n\right) =$$

$$= \prod_{(i,j)\in\lambda} \frac{(i-j+n)\,(i-j+n+\eta+\eta'+\nu+\nu')\,(i-j+n+\eta+\nu)\,(i-j+n+\eta'+\nu)}{h^2_\lambda(i,j)\left(\lambda'_j+\mu_i-i-j+1+2n+\eta+\eta'+\nu+\nu'\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{(i-j-n)\,(i-j-n-\eta-\eta'-\nu-\nu')\,(i-j-n-\eta-\nu')\,(i-j-n-\eta'-\nu')}{h^2_\mu(i,j)\left(\lambda_i+\mu'_j-i-j+1-2n-\eta-\eta'-\nu-\nu'\right)^2}\,.$$

Note that individual conformal blocks in the sum over n in (5.6) give the corresponding terms in the Fredholm series of the ${}_2F_1$ kernel determinant. Numerical checks for randomly chosen η,η',ν,ν' show that the expansions (5.3) and (5.6) perfectly match for intermediate values of t. In particular, this confirms the conjectural expression (5.4).

5.1.2. Whittaker kernel. The Whittaker kernel [9, 10] emerges in the limit

$$K_{\mathrm{W}}(x,y) = \lim_{\eta'\to\infty} \frac{1}{\eta'}\, K_{\mathrm{G}}\left(1-\frac{x}{\eta'}, 1-\frac{y}{\eta'}\right).$$

It contains three parameters ν, ν', η and has integrable form (5.1), λ is the same as above and

$$\varphi_{\mathrm{W}}(x) = \Gamma(1+\eta+\nu)\, x^{-\frac{1}{2}} W_{-\frac{\nu+\nu'+2\eta}{2}+\frac{1}{2}, \frac{\nu-\nu'}{2}}(x),$$
$$\psi_{\mathrm{W}}(x) = \Gamma(1+\eta+\nu')\, x^{-\frac{1}{2}} W_{-\frac{\nu+\nu'+2\eta}{2}-\frac{1}{2}, \frac{\nu-\nu'}{2}}(x),$$

where $W_{k,m}(x)$ denote the Whittaker functions.

Fredholm determinant

$$D_{\mathrm{W}}(t) = \det\left(1 - K_{\mathrm{W}}|_{(t,\infty)}\right), \qquad t \in (0,\infty),$$

is related to a particular Painlevé V tau function by

$$D_{\mathrm{W}}(t) = t^{-\frac{(\nu-\nu')^2}{4}} \tau_{\mathrm{V}}(t), \tag{5.7}$$
$$(\theta_0, \theta_t, \theta_*)_{\mathrm{V}} = \frac{1}{2}(\nu-\nu', 0, 2\eta+\nu+\nu'). \tag{5.8}$$

Its expansion around $t=0$ may be found from

$$D_{\mathrm{W}}(t) = \lim_{\eta'\to\infty} D_{\mathrm{G}}\Big(1 - \frac{t}{\eta'}\Big).$$

Namely, the appropriate termwise limit of (5.3) gives

$$D_{\mathrm{W}}(t) = \sum_{n\in\mathbb{Z}} C_{\mathrm{W}}(\nu+n, \nu'+n, \eta-n)\, t^{(\nu+n)(\nu'+n)} \mathcal{B}_{\mathrm{W}}(\nu+n, \nu'+n, \eta-n; t), \tag{5.9}$$

where the limits of structure constants and conformal blocks are

$$C_{\mathrm{W}}(\nu,\nu',\eta) = G\left[\begin{array}{c} 1+\nu, 1-\nu, 1+\nu', 1-\nu', 1+\eta, 1+\eta+\nu+\nu' \\ 1+\nu+\nu', 1-\nu-\nu', 1+\eta+\nu, 1+\eta+\nu' \end{array}\right],$$
$$\mathcal{B}_{\mathrm{W}}(\nu,\nu',\eta;t) = \mathcal{B}_{\mathrm{V}}\left(\frac{\nu-\nu'}{2}, 0, \eta+\frac{\nu+\nu'}{2}, \frac{\nu+\nu'}{2}; t\right),$$

and $\mathcal{B}_{\mathrm{V}}$ was defined in (4.15)–(4.16). Although we are not able to write similar combinatorial expansion at $t=\infty$, in the latter case $D_{\mathrm{W}}(t)$ can still be expanded into Fredholm series. Hence, for example,

$$D_{\mathrm{W}}(t\to\infty) = 1 - \lambda\,\Gamma(1+\eta+\nu)\,\Gamma(1+\eta+\nu')\, e^{-t} t^{-(2+2\eta+\nu+\nu')}\left[1 + O\left(t^{-1}\right)\right].$$

5.1.3. Confluent hypergeometric kernel. Another interesting scaling limit of the ${}_2F_1$ kernel corresponds to setting

$$\nu' = \nu'_0 - i\Lambda, \qquad \eta = \eta_0 + i\Lambda,$$

and then considering

$$K_{\mathrm{F}}(x,y) = \lim_{\Lambda\to\infty} \frac{1}{\Lambda} K_{\mathrm{G}}\left(\frac{x}{\Lambda}, \frac{y}{\Lambda}\right).$$

The result is the so-called confluent hypergeometric kernel [10, 15]. It depends on three parameters

$$r_+ = \nu + \eta', \qquad r_- = \nu_0' + \eta_0, \qquad \xi = \frac{1 - e^{2\pi i \nu}}{2\pi} \, e^{\frac{i\pi(r_- - r_+)}{2}},$$

and is given by (5.1) with

$$\begin{aligned}
&\lambda = \xi \, \Gamma\left[\begin{array}{c} 1 + r_+, 1 + r_- \\ 1 + r_+ + r_-, 2 + r_+ + r_- \end{array} \right], \\
&\varphi_{\mathrm{F}}(x) = x^{1 + \frac{r_+ + r_-}{2}} e^{-\frac{ix}{2}} {}_1F_1\left(r_+ + 1, r_+ + r_- + 2, ix\right), \\
&\psi_{\mathrm{F}}(x) = x^{\frac{r_+ + r_-}{2}} e^{-\frac{ix}{2}} {}_1F_1\left(r_+, r_+ + r_-, ix\right).
\end{aligned}$$

Similarly to (5.7)–(5.8), the ${}_1F_1$ kernel determinant

$$D_{\mathrm{F}}(t) = \det\left(1 - K_{\mathrm{F}}\big|_{(0,t)}\right), \qquad t \in (0, \infty),$$

can be expressed [10] in terms of a Painlevé V tau function:

$$D_{\mathrm{F}}(t) = t^{-\frac{(r_+ + r_-)^2}{4}} \tau_{\mathrm{V}}(it), \tag{5.10}$$

$$\left(\theta_0, \theta_t, \theta_*\right)_{\mathrm{V}} = \frac{1}{2}\left(r_+ + r_-, 0, r_+ - r_-\right). \tag{5.11}$$

Note that $D_{\mathrm{F}}(t) = \lim_{\Lambda \to \infty} D_{\mathrm{G}}\left(\frac{t}{\Lambda}\right)$. Applying this termwise to (5.6) and using the properties (A.1)–(A.2) of the Barnes function, we derive the expansion of $D_{\mathrm{F}}(t)$ at $t = 0$:

$$D_{\mathrm{F}}(t) = \sum_{n=0}^{\infty} C_{\mathrm{F}}\left(r_+, r_-, n\right) (-\xi)^n \, t^{n(n + r_+ + r_-)} \sum_{\lambda, \mu \in \mathbb{Y} | \lambda_1, \mu_1' \leq n} \mathcal{B}^{\mathrm{F}}_{\lambda,\mu}\left(r_+, r_-, n\right) (it)^{|\lambda| + |\mu|}, \tag{5.12}$$

where

$$\begin{aligned}
&C_{\mathrm{F}}\left(r_+, r_-, n\right) = G\left[\begin{array}{c} 1 + n, 1 + r_+ + r_- + n \\ 1 + r_+ + r_- + 2n \end{array} \right]^2 G\left[\begin{array}{c} 1 + r_+ + n, 1 + r_- + n \\ 1 + r_+, 1 + r_- \end{array} \right], \\
&\mathcal{B}^{\mathrm{F}}_{\lambda,\mu}\left(r_+, r_-, n\right) = \prod_{(i,j) \in \lambda} \frac{(i - j + n)(i - j + n + r_+)(i - j + n + r_+ + r_-)}{h_\lambda^2(i,j)\left(\lambda_j' + \mu_i - i - j + 1 + 2n + r_+ + r_-\right)^2} \times \\
&\times \prod_{(i,j) \in \mu} \frac{(i - j - n)(i - j - n - r_-)(i - j - n - r_+ - r_-)}{h_\mu^2(i,j)\left(\lambda_i + \mu_j' - i - j + 1 - 2n - r_+ - r_-\right)^2}.
\end{aligned}$$

5.1.4. Sine kernel. Certain specializations of the ${}_1F_1$ kernel play an important role in random matrix theory. In particular, for $r_+ = r_- = r$ it coincides with the Bessel kernel [40, 59]

$$K_{\mathrm{B}}(x, y) = \frac{\pi \xi \sqrt{xy}}{2} \frac{J_{r + \frac12}\left(\frac{x}{2}\right) J_{r - \frac12}\left(\frac{y}{2}\right) - J_{r - \frac12}\left(\frac{x}{2}\right) J_{r + \frac12}\left(\frac{y}{2}\right)}{x - y},$$

which in the case $r = 0$ reduces to the celebrated sine kernel

$$K_{\mathrm{sine}}(x, y) = \frac{2\xi \sin\frac{x - y}{2}}{x - y}.$$

It is well-known that the determinant

$$D_{\text{sine}}(t) = \det\left(1 - K_{\text{sine}}|_{(0,t)}\right)$$

for $\xi = \frac{1}{2\pi}$ coincides with the scaled gap probability in the bulk of the Gaussian Unitary Ensemble [23]. The expansion (5.12) thus gives a complete series for this quantity:

$$D_{\text{sine}}(t) = \sum_{n=0}^{\infty} \frac{G^6(1+n)}{G^2(1+2n)} (-\xi)^n t^{n^2} \sum_{\lambda,\mu\in\mathbb{Y}|\lambda_1,\mu_1'\leq n} \mathcal{B}^{\text{sine}}_{\lambda,\mu}(n)\,(it)^{|\lambda|+|\mu|}, \tag{5.13}$$

where

$$\mathcal{B}^{\text{sine}}_{\lambda,\mu}(n) = \prod_{(i,j)\in\lambda} \frac{(i-j+n)^3}{h_\lambda^2(i,j)\left(\lambda_j' + \mu_i - i - j + 1 + 2n\right)^2} \times$$
$$\times \prod_{(i,j)\in\mu} \frac{(i-j-n)^3}{h_\mu^2(i,j)\left(\lambda_i + \mu_j' - i - j + 1 - 2n\right)^2}.$$

First terms of the series (5.13) are recorded in the Appendix B. In particular, they reproduce the results obtained by an iterative expansion of the corresponding Painlevé V solution, cf Eq. (8.114) in [23]. Note that our $t = 2\pi t_{[23]}$, $\xi = \frac{\xi_{[23]}}{2\pi}$. We have also checked the agreement of (5.13) with the known large gap ($t \to \infty$) asymptotics [18]

$$D_{\text{sine}}(4t)\Big|_{\xi=\frac{1}{2\pi}} = \sqrt{\pi}\, G^2\left(\frac{1}{2}\right) t^{-\frac{1}{4}} e^{-\frac{t^2}{2}} \left[1 + \frac{1}{32}t^{-2} + \frac{81}{2048}t^{-4} + O\left(t^{-6}\right)\right].$$

5.1.5. Modified Bessel kernel. One may also study a further scaling limit of the ${}_1F_1$ kernel by setting

$$r_\pm = \frac{r}{2} \mp i\Lambda, \qquad \xi = \xi_{\text{B2}} \frac{re^{\pi\Lambda}}{2\pi},$$

and defining

$$K_{\text{B2}}(x,y) = \lim_{\Lambda\to\infty} \frac{1}{\Lambda} K_{\text{F}}\left(\frac{x}{\Lambda}, \frac{y}{\Lambda}\right).$$

Asymptotic properties of the confluent hypergeometric function imply that

$$K_{\text{B2}}(x,y) = \xi_{\text{B2}}\sqrt{xy}\, \frac{I_{r+1}\left(2\sqrt{x}\right) I_{r-1}\left(2\sqrt{y}\right) - I_{r-1}\left(2\sqrt{x}\right) I_{r+1}\left(2\sqrt{y}\right)}{x-y}.$$

Fredholm determinant $D_{\text{B2}}(t) = \det\left(1 - K_{\text{B2}}|_{(0,t)}\right)$ is related to a tau function of Painlevé III_1' with $\theta_* = -\theta_\star = \frac{r}{2}$ by

$$D_{\text{B2}}(t) = t^{-\frac{r^2}{4}} e^{\frac{t}{2}} \tau_{\text{III}_1'}(t). \tag{5.14}$$

Its small gap expansion can be calculated using that $D_{\text{B2}}(t) = \lim_{\Lambda\to\infty} D_{\text{F}}\left(\frac{t}{\Lambda}\right)$. We find

$$D_{\text{B2}}(t) = \sum_{n=0}^{\infty} G\left[\begin{array}{c} 1+n, 1+r+n \\ 1+r+2n \end{array}\right]^2 (-\xi_{\text{B2}} r)^n t^{n(n+r)} \sum_{\lambda,\mu\in\mathbb{Y}|\lambda_1,\mu_1'\leq n} \mathcal{B}^{\text{B2}}_{\lambda,\mu}(r,n)\, t^{|\lambda|+|\mu|}, \tag{5.15}$$

$$\mathcal{B}^{\mathrm{B2}}_{\lambda,\mu}(r,n) = \prod_{(i,j)\in\lambda} \frac{(i-j+n)(i-j+n+r)}{h_\lambda^2(i,j)\left(\lambda'_j+\mu_i-i-j+1+2n+r\right)^2} \times$$
$$\times \prod_{(i,j)\in\mu} \frac{(i-j-n)(i-j-n-r)}{h_\mu^2(i,j)\left(\lambda_i+\mu'_j-i-j+1-2n-r\right)^2}.$$

5.2. Sine-Gordon exponential fields

A well-known example of appearance of Painlevé transcendents in integrable QFT is provided by the two-point correlation function of exponential fields $Q(mr) = \langle \mathcal{O}_\nu(0)\,\mathcal{O}_{\nu'}(r)\rangle$ in the sine-Gordon model at the free-fermion point [7, 51]. The spectrum of this model consists of fermionic excitations of mass m, parameterized by the topological charge $\epsilon = \pm 1$ and rapidity $\theta \in \mathbb{R}$. Lattice counterparts of the exponential fields have been introduced and studied in [28, 47].

5.2.1. From form factors to Macdonald kernel. Under normalization $\langle \mathcal{O}_\nu \rangle = 1$, the exponential fields are completely determined by their two-particle form factors [50]

$$\mathcal{F}_\nu(\theta,\theta') = {}^{+-}\langle \theta;\theta'|\mathcal{O}_\nu(0)|vac\rangle = \frac{i\sin\pi\nu}{2\pi}\frac{e^{\nu(\theta'-\theta)}}{\cosh\frac{\theta'-\theta}{2}}.$$

Multiparticle form factors can be written as determinants of two-particle ones. This allows to sum up the form factor expansion

$$Q(mr) = \sum_{n=0}^{\infty}\sum_{\epsilon_1,\ldots,\epsilon_n=\pm}\frac{1}{n!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} d\theta_1\ldots d\theta_n\, e^{-mr\sum_{k=1}^n\cosh\theta_k}\times$$
$$\times \langle vac|\mathcal{O}_\nu(0)|\theta_1,\ldots,\theta_n\rangle_{\epsilon_1,\ldots,\epsilon_n}{}^{\epsilon_1,\ldots,\epsilon_n}\langle\theta_1,\ldots,\theta_n|\mathcal{O}_{\nu'}(0)|vac\rangle$$

to Fredholm determinant $Q(mr) = \det(1-K_{\mathrm{SG}})$. The corresponding kernel acts on $L^2(\mathbb{R})$ and is expressed in terms of dressed two-particle form factors:

$$K_{\mathrm{SG}}(\theta,\theta') = \int_{-\infty}^{\infty}\mathcal{F}_{-\nu}(\theta'',\theta)\,\mathcal{F}_{\nu'}(\theta'',\theta')\,e^{-\frac{mr}{2}(\cosh\theta+2\cosh\theta''+\cosh\theta')}d\theta''.$$

Let us show that $K_{\mathrm{SG}}(\theta,\theta')$ is equivalent to a more familiar classical integrable kernel $K_{\mathrm{M}}(x,y)$ on $L^2(\frac{m^2r^2}{4},\infty)$. The latter is defined by (5.1) with $\lambda = \pi^{-2}\sin\pi\nu\sin\pi\nu'$ and φ, ψ given by Macdonald functions

$$\varphi_{\mathrm{M}}(x) = 2\sqrt{x}K_{\nu'-\nu+1}\left(2\sqrt{x}\right),\qquad \psi_{\mathrm{M}}(x) = 2K_{\nu'-\nu}\left(2\sqrt{x}\right). \tag{5.16}$$

This kernel can be seen as a further scaling limit of the Whittaker kernel from the previous subsection. Indeed, one may check that

$$K_{\mathrm{M}}(x,y) = \lim_{\eta\to\infty}\frac{1}{\eta}K_{\mathrm{W}}\left(\frac{x}{\eta},\frac{y}{\eta}\right).$$

By equivalence of K_{SG} and K_{M} we mean that $\mathrm{Tr}\,K_{\mathrm{SG}}^n = \mathrm{Tr}\,K_{\mathrm{M}}^n$ for any $n\in\mathbb{Z}_{\geq 0}$.

Here is a proof. First note that the Macdonald kernel admits an alternative simple form

$$K_{\mathrm{M}}(x,y) = \lambda \int_1^\infty \psi_{\mathrm{M}}(xt)\psi_{\mathrm{M}}(yt)\,dt. \tag{5.17}$$

This representation results from the identity

$$\frac{d}{dt}\left[\varphi_{\mathrm{M}}(xt)\psi_{\mathrm{M}}(yt) - \varphi_{\mathrm{M}}(yt)\psi_{\mathrm{M}}(xt)\right] = -(x-y)\,\psi_{\mathrm{M}}(xt)\psi_{\mathrm{M}}(yt),$$

which is itself an easy consequence of the differentiation formulas

$$x\frac{d}{dx}\left(\begin{array}{c}\varphi_{\mathrm{M}}(x)\\ \psi_{\mathrm{M}}(x)\end{array}\right) = \left(\begin{array}{cc}\frac{\nu-\nu'}{2} & -x\\ -1 & \frac{\nu'-\nu}{2}\end{array}\right)\left(\begin{array}{c}\varphi_{\mathrm{M}}(x)\\ \psi_{\mathrm{M}}(x)\end{array}\right).$$

On the other hand, parameterizing the rapidities as $u = e^\theta$, one can write $\kappa_n = \operatorname{Tr} K^n_{\mathrm{SG}}$ as

$$\kappa_n = \lambda^n \int_0^\infty \ldots \int_0^\infty du_1 \ldots du_{2n} \prod_{j=1}^{n} \frac{u_{2j-1}^{\nu'-\nu}}{u_{2j}^{\nu'-\nu}} \prod_{j=1}^{2n} \frac{\exp\left\{-\frac{mr}{2}\left(u_j + u_j^{-1}\right)\right\}}{u_j + u_{j+1}}, \tag{5.18}$$

with $u_{2n+1} = u_1$. Now make in (5.18) the following replacements:

$$\frac{e^{-\frac{mr}{2}(u_{2j-1}+u_{2j})}}{u_{2j-1}+u_{2j}} = \int_{\frac{mr}{2}}^\infty e^{-t_{2j-1}(u_{2j-1}+u_{2j})}dt_{2j-1},$$

$$\frac{e^{-\frac{mr}{2}\left(u_{2j}^{-1}+u_{2j+1}^{-1}\right)}}{u_{2j-1}+u_{2j}} = u_{2j}^{-1}u_{2j+1}^{-1}\int_{\frac{mr}{2}}^\infty e^{-t_{2j}\left(u_{2j}^{-1}+u_{2j+1}^{-1}\right)}dt_{2j-1},$$

where $j = 1,\ldots,n$. This yields a $4n$-fold integral

$$\int_0^\infty \frac{du_1}{u_1}\ldots\int_0^\infty \frac{du_{2n}}{u_{2n}} \int_{\frac{mr}{2}}^\infty dt_1 \ldots \int_{\frac{mr}{2}}^\infty dt_{2n} \prod_{j=1}^{n} \frac{u_{2j-1}^{\nu'-\nu}}{u_{2j}^{\nu'-\nu}}\, e^{-t_{2j-1}u_{2j-1}-t_{2j-2}u_{2j-1}^{-1}-t_{2j-1}u_{2j}-t_{2j}u_{2j}^{-1}},$$

with $t_0 = t_{2n}$. The variables $u_1,\ldots,u_{2n}$ are now decoupled. Integrating them out with the help of the standard integral representation of the Macdonald function

$$\int_0^\infty u^{-1\pm(\nu'-\nu)}e^{-tu-t'u^{-1}}du = (t/t')^{\mp\frac{\nu'-\nu}{2}}\,\psi_{\mathrm{M}}(tt'), \qquad t,t'>0,$$

we finally obtain

$$\kappa_n = \lambda^n \int_{\frac{mr}{2}}^\infty dt_1 \ldots \int_{\frac{mr}{2}}^\infty dt_{2n} \prod_{j=1}^{2n} \psi_{\mathrm{M}}(t_{j-1}t_j).$$

After the change of variables $t_{2j-1} \mapsto \frac{mr}{2}t_{2j-1}$, $t_{2j} \mapsto \frac{2}{mr}t_{2j}$ the last expression can obviously be written as $\operatorname{Tr} K^n_{\mathrm{M}}$ with K_{M} given by (5.17).

5.2.2. Painlevé III and asymptotics. Painlevé representations of the two-point function of exponential fields [7, 51] can now be rederived by applying the standard random matrix theory techniques [56] to the Macdonald kernel. The final result is that $Q(mr) = \det\left(1 - K_{\text{M}}\big|_{\left(\frac{m^2r^2}{4},\infty\right)}\right)$ coincides, up to a simple prefactor, with a tau function of Painlevé III_1' equation with parameters $\theta_* = -\theta_\star = \frac{\nu-\nu'}{2}$:

$$Q(2\sqrt{t}) = t^{-\frac{(\nu-\nu')^2}{4}} e^{\frac{t}{2}} \tau_{\text{III}_1'}(t)\,.$$

The integration constants specifying this tau function are [31]

$$\sigma = \frac{\nu+\nu'}{2}, \qquad s_{\text{III}_1'} = 1. \tag{5.19}$$

In general, the tau function is defined up to multiplication by a constant, which in the case at hand is fixed by normalization of the VEVs: $Q(mr) \simeq 1$ as $r \to \infty$. Subleading corrections to this long-distance behaviour can be obtained from the form factor expansion. For instance, taking into account the contribution of two-particle states, we find

$$1 - Q(2\sqrt{t}) = \lambda \underbrace{\int_t^\infty \Big(\varphi_{\text{M}}'(x)\psi_{\text{M}}(x) - \varphi_{\text{M}}(x)\psi_{\text{M}}'(x)\Big)\,dx}_{O\left(t^{-1/2}e^{-4\sqrt{t}}\right)} + O\left(t^{-1}e^{-8\sqrt{t}}\right).$$

Short-distance asymptotics of $Q(mr)$ is also known. Assume that $|\text{Re}\,(\nu+\nu')\,| < 1$, then, as $t \to 0$,

$$Q(2\sqrt{t}) \simeq C_{\text{SG}}\,(\nu,\nu')\, t^{\nu\nu'}.$$

The value of σ in (5.19) is determined by the exponent $\nu\nu'$, found in [31]. The coefficient $C_{\text{SG}}\,(\nu,\nu')$ was calculated by Basor and Tracy in [3]:

$$C_{\text{SG}}\,(\nu,\nu') = G\left[\begin{array}{c} 1+\nu, 1-\nu, 1+\nu', 1-\nu' \\ 1+\nu+\nu', 1-\nu-\nu' \end{array}\right]. \tag{5.20}$$

Note that the last expression coincides with $C_{\text{III}_1'}\left(\frac{\nu-\nu'}{2}, \frac{\nu'-\nu}{2}, \frac{\nu+\nu'}{2}\right)$ defined by (4.20). This simply means that the normalization of $\tau_{\text{III}_1'}(t)$ in Conjecture 4 corresponds to setting $\langle \mathcal{O}_\nu \rangle = 1$ in the sine-Gordon case.

5.2.3. Short-distance expansion of $\langle \mathcal{O}_\nu\,(0)\,\mathcal{O}_{\nu'}(r)\rangle$. We are now ready to write complete short-distance expansion of the two-point correlator $Q(mr)$. Combining the above with Conjecture 4 gives the following series:

$$Q(mr) = \sum_{n\in\mathbb{Z}} C_{\text{SG}}(\nu+n, \nu'+n) \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{\text{SG}}_{\lambda,\mu}\,(\nu+n,\nu'+n) \left(\frac{m^2r^2}{4}\right)^{(\nu+n)(\nu'+n)+|\lambda|+|\mu|}, \tag{5.21}$$

where $C_{\text{SG}}(\nu, \nu')$ is defined by (5.20) and

$$\mathcal{B}^{\text{SG}}_{\lambda,\mu}(\nu, \nu') = \prod_{(i,j)\in\lambda} \frac{(i-j+\nu)(i-j+\nu')}{h^2_\lambda(i,j)\left(\lambda'_j+\mu_i-i-j+1+\nu+\nu'\right)^2} \times$$
$$\times \prod_{(i,j)\in\mu} \frac{(i-j-\nu)(i-j-\nu')}{h^2_\mu(i,j)\left(\lambda_i+\mu'_j-i-j+1-\nu-\nu'\right)^2}.$$

The series (5.21) has a familiar structure of conformal perturbation expansion [19, 62]. The non-analytic factors $m^{2(\nu+n)(\nu'+n)}$ correspond to non-perturbative VEVs of the primary fields which appear in the operator product expansion $\mathcal{O}_\nu(0)\,\mathcal{O}_{\nu'}(r)$. All other corrections, including the VEVs of descendant fields and CPT, come in integer powers of the coupling m^2.

Fig. 5 illustrates how well the series (5.21) fits form factor expansion to give all-distance behaviour of the correlator. We fix $\nu = 0.3$, $\nu' = 0.45$ and compute the expansion $Q_{15}(mr)$ taking into account the terms with $n = -4, \ldots, 4$ up to descendant level 15, as we did before for P_{VI}. Plots (A), (B), (C), (D) correspond to the logarithms of $-\sum_{j=1}^{\ell-1}\frac{1}{j}\operatorname{Tr}K^j_{\text{SG}} - \ln Q_{15}(mr)$ (solid lines) and $\frac{1}{\ell}\operatorname{Tr}K^\ell_{\text{SG}}$ (dotted lines) for $\ell = 1, 2, 3, 4$. Hence $Q_{15}(mr)$ correctly accounts for the 2-particle form factor contribution to long-distance asymptotics up to $mr \approx 3.2$, 4-particle contribution up to $mr \approx 2.6$, 6-particle and 8-particle ones up to $mr \approx 2.1$ and $mr \approx 1.7$.

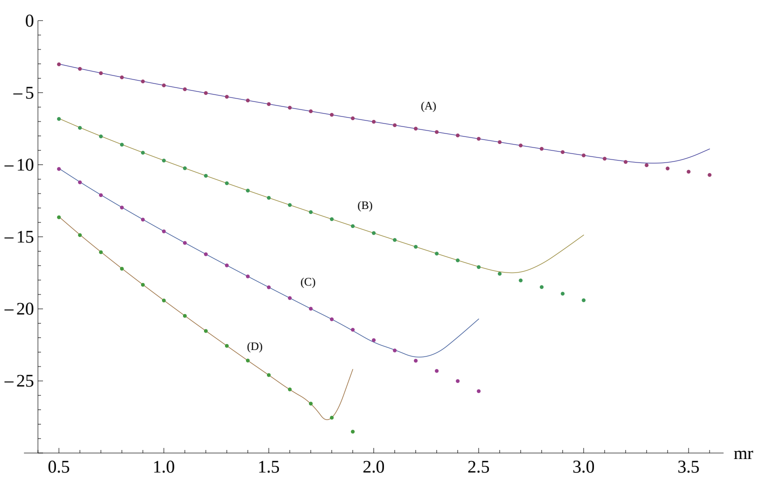

Fig. 5: Short-distance vs form factor expansion

Let us stress that we are dealing here with a correlation function of *massive* theory. Its description by *holomorphic* conformal blocks therefore looks rather surprising and is presumably related to the affine $\widehat{sl(2)}$ symmetry of the free-fermion sine-Gordon theory [33].

5.2.4. Painlevé III and 2D polymers. The change of variables $q_{\mathrm{III}'_1}(t) = \frac{r}{4}\exp\psi(r)$, $r = 4\sqrt{t}$ maps $P_{\mathrm{III}'_1}$ with $\theta_\star = 0$, $\theta_* = \frac12$ to radial sinh-Gordon equation

$$\psi'' + \frac{1}{r}\psi' = \frac12 \sinh 2\psi. \tag{5.22}$$

A particular solution of this equation describes universal scaling functions of 2D polymers [20, 63]. It is characterized by the boundary conditions

$$\psi(r\to 0) \sim -\frac13 \ln r - \frac12 \ln\frac{\mu}{4} + O\left(r^{4/3}\right), \qquad \mu = \frac{\Gamma^2(1/3)}{\Gamma^2(2/3)}, \tag{5.23}$$

which in our notation correspond to integration constants $\sigma_{\mathrm{III}'_1} = \frac16$, $s_{\mathrm{III}'_1} = 1$. In fact the relevant $P_{\mathrm{III}'_1}$ solution is a Bäcklund transform of a solution associated to the tau function considered in the previous subsection. The precise relation between the two quantities is

$$\sinh^2\psi(r) = -\left[(\ln Q(r))'' + r^{-1}(\ln Q(r))'\right]_{\nu=\nu'=\frac16}.$$

On the other hand, it is known [46] that $P_{\mathrm{III}'_1}$ with $\theta_\star = 0$, $\theta_* = \frac12$ (and hence the radial sinh-Gordon equation!) is equivalent to $P_{\mathrm{III}'_3}$. Namely, if we set

$$t_{\mathrm{III}'_3} = \frac{t^2}{16}, \qquad q_{\mathrm{III}'_3}\left(t_{\mathrm{III}'_3}\right) = \frac{q^2(t)}{4},$$

then $q(t)$ satisfies appropriate $P_{\mathrm{III}'_1}$. This allows to give an alternative characterization of the solution (5.23) via the expansion (4.25)–(4.28):

$$e^{-2\psi(r)} = -4r^{-1}\frac{d}{dr}r\frac{d}{dr}\ln\tau_{\mathrm{III}'_3}\left(2^{-12}r^4\right)\Big|_{s_{\mathrm{III}'_3}=1,\sigma=\frac16},$$

or, in yet another form,

$$e^{\psi(r)} = \frac{\tau_{\mathrm{III}'_3}\left(2^{-12}r^4\right)\Big|_{s_{\mathrm{III}'_3}=1,\sigma=\frac16}}{\tau_{\mathrm{III}'_3}\left(2^{-12}r^4\right)\Big|_{s_{\mathrm{III}'_3}=1,\sigma=\frac13}},$$

where the normalization of both tau functions in the last formula is precisely the same as in Conjecture 6.

6. Discussion

We believe that by explaining the title of this paper we have partially answered P. Deift's question from the Introduction. Besides the obvious need for rigorous proofs of our claims in Section 4, many other questions beg to be addressed. Why instantons? Is there a way to obtain irregular "form factor" expansions at ∞ for general solutions of P_{V} and P_{III}? What about P_{IV}, P_{II} and P_{I}?

A particularly interesting problem, already mentioned above, concerns the computation of connection coefficients of Painlevé tau functions (akin to Dyson-Widom constants in random matrix theory). In the P_{VI} case this is very much related to determining the fusion matrix for $c = 1$ generic conformal blocks.

Another intriguing issue is the quantization of Painlevé equations [41, 60]. The existing paradigm usually associates isomonodromic deformations to semiclassical limit of CFT [53]. For instance, the scalar Lax pairs for $P_{\text{VI}-\text{I}}$ emerge in the $c \to \infty$ limit of two BPZ-type differential operators [42]. The results presented here and in [27] suggest a completely different, $c = 1$ point of view. In this picture, classical Riccati solutions of Painlevé equations may be naturally deformed to Coulomb β-integrals. It would be nice to understand whether the general case allows for a similar β-deformation.

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Appendix A. Barnes function

Barnes G-function satisfies the functional equation $G\,(1+z) = \Gamma\,(z)\,G\,(z)$ and is defined as the infinite product

$$G\,(1+z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2\,(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right),$$

where γ is the Euler's constant, or via the integral representation

$$G(1+z) = (2\pi)^{\frac{z}{2}} \exp \int_0^{\infty} \frac{dt}{t} \left[\frac{1 - e^{-zt}}{4\sinh^2 \frac{t}{2}} - \frac{z}{t} + \frac{z^2}{2}\, e^{-t}\right], \qquad \operatorname{Re} z > -1.$$

It is analytic in the whole complex plane and has the following asymptotic expansion as $|z| \to \infty$, $\arg z \neq \pi$:

$$\ln G(1+z) = \left(\frac{z^2}{2} - \frac{1}{12}\right) \ln z - \frac{3z^2}{4} + \frac{z}{2} \ln 2\pi + \zeta'(-1) + O\left(z^{-2}\right).$$

One of the consequences of this asymptotic behaviour is the formula

$$G\left[\begin{array}{c} 1+z+\alpha, 1+z-\alpha \\ 1+z+\beta, 1+z-\beta \end{array}\right] = z^{\alpha^2 - \beta^2} \left[1 + O\left(z^{-2}\right)\right]. \tag{A.1}$$

Another useful relation is

$$G\left[\begin{array}{c} 1+z+n, 1-z \\ 1-z-n, 1+z \end{array}\right] = (-1)^{\frac{n(n+1)}{2}} \left(\frac{\pi}{\sin \pi z}\right)^n, \qquad n \in \mathbb{Z}. \tag{A.2}$$

It is easy to deduce from it that, as $\varepsilon \to 0$,

$$G\left(1+\varepsilon-n\right) \sim \varepsilon^n \left(-1\right)^{\frac{n(n-1)}{2}} G\left(1+n\right), \qquad n \in \mathbb{Z}_{\geq 0}. \tag{A.3}$$

Appendix B. Sine kernel conformal blocks

Consider the functions

$$\mathcal{B}_{\text{sine}}(n;t) = \sum_{\lambda,\mu \in \mathbb{Y} | \lambda_1, \mu_1' \leq n} \mathcal{B}^{\text{sine}}_{\lambda,\mu}(n) \, (it)^{|\lambda|+|\mu|},$$

which appear in the expansion (5.13) of the GUE gap probability. Below we record the terms contributing to $D_{\text{sine}}(t)$ as at least t^{30}:

$$\mathcal{B}_{\text{sine}}(0;t) = \mathcal{B}_{\text{sine}}(1;t) = 1,$$

$$\begin{aligned}
\mathcal{B}_{\text{sine}}(2;t) = 1 &- \frac{t^2}{75} + \frac{t^4}{7840} - \frac{t^6}{1134000} + \frac{t^8}{219542400} - \frac{t^{10}}{55091836800} + \frac{t^{12}}{17435658240000} \\
&- \frac{t^{14}}{6802522062336000} + \frac{t^{16}}{3210079038566400000} - \frac{t^{18}}{1803084500809912320000} + \\
&+ \frac{t^{20}}{1189192769988708925440000} - \frac{t^{22}}{910206422681575219200000000} + \\
&+ \frac{t^{24}}{800331904605748883816448000000} - \frac{t^{26}}{801284680682660489630515200000000} + O\left(t^{28}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{\text{sine}}(3;t) = 1 &- \frac{18\,t^2}{1225} + \frac{t^4}{8820} - \frac{2293\,t^6}{3922033500} + \frac{3581\,t^8}{1616027212800} - \frac{71\,t^{10}}{10908183686400} + \\
&+ \frac{94789\,t^{12}}{617883156732441600} - \frac{76477\,t^{14}}{2570452778021883955200} \\
&+ \frac{407221\,t^{16}}{8412390909889802035200000} - \frac{245265109\,t^{18}}{36550901363123828118997278720000} \\
&+ \frac{40956413\,t^{20}}{510254748374093327017340928000000} + O\left(t^{22}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{\text{sine}}(4;t) = 1 &- \frac{20\,t^2}{1323} + \frac{83\,t^4}{711480} - \frac{174931\,t^6}{286339821768} + \frac{9605\,t^8}{3926946127104} \\
&- \frac{4585051\,t^{10}}{572172412994582400} + \frac{5892151877\,t^{12}}{262340410524913476467712} \\
&- \frac{586063249\,t^{14}}{105560784235024708388198400} + O\left(t^{16}\right),
\end{aligned}$$

$$\mathcal{B}_{\text{sine}}(5;t) = 1 - \frac{50\,t^2}{3267} + \frac{475\,t^4}{4008004} + O\left(t^6\right).$$

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